The Second Workshop on Digitalization and Computable Models

(WDCM-2020, July 20-24, 2020, Novosibirsk/online-meeting)

Abstracts of Invited Talks

Complexity of fixed-point selection functions

M.M. Arslanov (Kazan Federal University)

A version of Kleene's recursion theorem states that the fixed points of a uniformly computable sequence of functions can be found in a uniformly computable way. In my earlier works I extended Kleene's recursion theorem in several ways. In particular, for any c.e. set A, if $\emptyset' \not\leq_T A$, then all computable in A functions have fixed-points. In which cases a uniform version of the recursion theorem holds for such classes of functions? In my talk I will give an overview on recent investigations of these and related questions.

Arman Darbinyan

(Texas A&M University, College Station, TX, USA)

Computable groups and computable group orderings

An important class of abstract groups is the one that consists of linearly ordered groups whose orders are invariant under left (and right) group multiplications. From computability point of view it is interesting to investigate when orderable groups admit computable orders. In particular, a question of Downey and Kurtz asks about existence of computable orderable groups that do not admit computable orders with respect to any group presentation. In my talk I will discuss how to construct such examples of bi-orderable groups.

Rumen Dimitrov

Recent Results in Cohesive Powers

While studying the structure of the lattice $\mathcal{L}^*(V_\infty)$ of computably enumerable subspaces of V_∞ modulo finite dimension we encountered interesting non-standard fields. These fields were used in [1] to characterize the principal filters of the equivalence classes of quasimaximal spaces with extendable bases. Cohesive powers of computable structures were formally introduced in [2] and used in [3] to classify the orbits of such quasimaximal spaces in $\mathcal{L}^*(V_\infty)$.

In this talk I will survey some recent results related to properties of cohesive powers and their automorphisms.

References

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Jun Le Goh

(University of Wisconsin–Madison, Madison, WI, USA)

A Sigma^1_1 axiom of finite choice and Steel forcing

Several variants of the Sigma^1_1 axiom of choice are known to be theories of hyperarithmetic analysis, i.e., all of their omega-models are closed under hyperarithmetic reduction and they hold in HYP(Y) for every subset Y of omega. Most theories of hyperarithmetic analysis are linearly ordered by provability, with the only known exception being the arithmetic Bolzano-Weierstrass theorem (Conidis). We introduce a Sigma^1_1 axiom of finite choice and use a variant of Steel forcing to show that it also lies to the side, strengthening results of Conidis.

RATIONALITY OF GROWTHS OF GROUPS

MENG-CHE "TURBO" HO

Abstract. For a group G, we know the word problem is solvable if and only if the group is computable. A problem dual to the word problem is finding a set of representatives for the group. We will consider the setting where these problems are solvable, and study their language theoretic complexity. In this setting, the word problem and the set of representatives have very different complexities.

If we further insist on the representatives to be geodesic, then the count $b(n)$ of representatives of length n gives the growth of the group. This is a geometric invariant of the group as $b(n)$ is the size of the n-ball in the Cayley graph of G. Gromov's polynomial growth theorem says that any group with $b(n)$ bounded by a polynomial is virtually nilpotent. One then wonders when is $b(n)$ precisely a polynomial, and this turns out to be related to the rationality of the growth of G and the set of representatives having low language theoretic complexity.

In this talk, we will survey past results on rationality of growths of groups. We also generalize past results and study the growths of the higher Baumslag-Solitar groups and torus bundle groups. If time permits, we will also discuss the rationality of the conjugacy growths of groups.

Parts of the talk are joint work with Seongjun Choi, Laura Ciobanu, Alex Evetts, Mark Pengitore, and Ayla P. Sanchez.

Date: June 22, 2020.

Patrick Lutz

(University of California, Berkeley, CA, USA)

Part 1 of Martin's Conjecture for Order Preserving Functions

Martin's conjecture is an attempt to make precise the idea that the only natural functions on the Turing degrees are the constant functions, the identity, and transfinite iterates of the Turing jump. The conjecture is typically divided into two parts. Very roughly, the first part states that every natural function on the Turing degrees is either eventually constant or eventually increasing and the second part states that the natural functions which are increasing form a well-order under eventual domination, where the successor operation in this well-order is the Turing jump.

In the 1980's, Slaman and Steel proved that the second part of Martin's conjecture holds for order-preserving Borel functions. We prove the complementary result that the first part of the conjecture also holds for order-preserving Borel functions. Our methods also yield several other new results, including an equivalence between the full part 1 of Martin's conjecture and a statement about the Rudin-Keisler order on ultrafilters on the Turing degrees.

In my talk, I will give an overview of Martin's conjecture and then describe our new results.

Russell Miller

Hilbert's Tenth Problem for Subrings of the Rational Numbers

For a subring R of the field $\mathbb{Q}$, *Hilbert's Tenth Problem* is the set

$$HTP(R) = \{f \in \mathbb{Z}[X_1, \ldots, X_n] \ : \ (\exists \vec{x} \in R^n)\ f(\vec{x}) = 0\}$$

of polynomial equations (in several variables) that have solutions in R. In 1970, Matiyasevich completed work by Davis, Putnam, and Robinson, showing that $HTP(\mathbb{Z})$ is undecidable, and indeed is computably isomorphic to the Halting Problem $\emptyset'$. It is an open question whether $HTP(\mathbb{Q})$ is also undecidable.

We will approach this open problem using subrings R of $\mathbb{Q}$. There is a bijection between the collection of such subrings and the power set of the set $\mathbb{P}$ of all prime numbers: each subset $W \subseteq \mathbb{P}$ corresponds to the ring $\mathbb{Z}[W^{-1}]$. This yields a topology on the space of subrings of $\mathbb{Q}$, homeomorphic to Cantor space $2^{\mathbb{P}}$.

We will prove theorems relating the behavior of $HTP(\mathbb{Q})$ to the behavior of $HTP(R)$ for "generic" subrings of $\mathbb{Q}$. The map sending W to $HTP(\mathbb{Z}[W^{-1}])$ does not preserve Turing reducibility, and sometimes even reverses it! We will also show that the situation for $\mathbb{Z}$ itself is unusual: for most subsets $W \subseteq \mathbb{P}$, the jump W' has no 1-reduction to the set $HTP(\mathbb{Z}[W^{-1}]$.

Portions of this work are joint with Ken Kramer, and a few parts are joint with Kirsten Eisenträger, Jennifer Park, and Alexandra Shlapentokh.

Benoit Monin

(Université Paris-Est Créteil, Créteil, France)

Reverse mathematics and the Ramsey's theorem for pairs

Every mathematician already heard his colleagues of professors say things like “Theorems A and B are equivalent” or “Theorem C does not follow from theorem D”. But what do we really mean by that? Indeed, from the viewpoint of classical logic, theorems are all pairwise equivalent within ZFC.

Reverse mathematics can be seen as an attempt to put a formal meaning on our intuition about this : two theorems are equivalent if each can be deduced from the another one, using in addition only a restricted set of axioms: the ones allowing to perform “elementary transformations”, that is, simply computations. The terminology itself comes from the used method to show axioms optimality: Once some axioms have been used to show a theorem T, we then try to demonstrate back these axioms from Theorem T, with the only help of the “ground axioms” allowing to perform elementary transformations : this is why these mathematics are “reversed”.

A simple observation that certainly contributed to the growth of reverse mathematics, is that they are very structured: Most mathematical theorems are equivalent to one among five axiomatic systems linearly ordered by strength: RCA0, WKL0, ACA0, ATR0 and Pi11-CA0. Among them RCA0 is the weakest which is also the one we always take for granted: it contains the basic axioms needed to develop computable mathematics. The systematic equivalences of theorems with one of these five systems led to them sharing the nickname “Big Five”.

One of the most studied theorems in reverse mathematics is the Ramsey theorem for pairs - RT22 - which is the cornerstone of a 50 year mathematical adventure. We will try to explain what is special about this theorem, by going through the history of its related reverse mathematical results, from the non-provability of RT22 within RCA0, to the separation between RT22 and SRT22 in omega-models.

Victor Selivanov

(Institute of Informatics Systems, Novosibirsk, Russia)

On computable fields of reals and some applications

In this joint work with Svetlana Selivanova we will discuss relationships of computably presentable ordered fields of reals with the field of computable reals. This has some applications to computable analysis (in the sense of TTE-approach), notably to linear algebra and to solving linear PDEs. Along with older results on general computability, we will mention more recent results about primitive recursive and polynomial-time computable versions of our approach.

Svetlana Selivanova

(KAIST, Daejeon, Republic of Korea)

Real Complexity Analysis Applied to Equations of Mathematical Physics

In this talk we briefly introduce the recently emerged and fast developing computability and complexity theories for real valued functions and operators, as well as provide examples of classifying different problems of analysis according to their algorithmic complexity. The considered framework provides a bridge between classical fields: “discrete” computability and complexity theories, from one side; and mathematical analysis (including functional spaces, differential equations etc.) from the other, to mutual benefit of both. As a main example we discuss recent achievements of real complexity analysis applied to equations of mathematical physics, including complexity classification of linear partial differential equations.

The Membership Problem in matrix semigroups

Pavel Semukhin
University of Oxford

Abstract. The *Membership Problem* for matrices is defined as follows: given a finite set of $n \times n$ matrices G and a matrix M, decide whether M belongs to the multiplicative semigroup generated by G. This problem was first shown to be undecidable for 6×6 matrices by A.A. Markov in 1940s and then for 3×3 matrices by M. Paterson in 1970.

On the other hand, in recent years, the Membership Problem was shown to be decidable in several special case of low-dimensional matrix semigroups. In this talk, we will give a brief history of previously known (un)decidability results and then discuss the newly discovered cases in which the Membership Problem becomes decidable.

CODING AND DECODING IN CLASSES OF STRUCTURES

ALEXANDRA SOSKOVA

Abstract

There are several ways to embed a class of structures to another class, as Borel embedding, Turing-computable embedding, etc. We always have a uniform method for coding each structure of one class in some structure of the other. For each of these reducibilities we have classes that are universal, in the sense that all other classes can be reduced to it. When a structure $\mathcal{A}$ from one class is coded effectively and uniformly in a structure $\mathcal{B}$ from another class, the question is: can we decode effectively and uniformly the structure $\mathcal{B}$ from $\mathcal{A}$? We are interested in cases where there is a uniform effective procedure for decoding, and in cases where the decoding is highly non-effective. We consider a notion of reducibility, introduced by Montalbán, based on the idea of effective interpretability between structures. It captures the idea of effective decoding. R. Miller proposed a notion of effective interpretability based on the notion of computable functor—a pair of Turing operators, the first one gives the Medvedev reduction and the second preserving the isomorphism, between copies. Harrison-Trainor, Melnikov, R. Miller, and Montalbán [2] prove that these the two notions of effective uniform interpretability coincide.

The class of undirected graphs and the class of linear orderings both lie on top under Turing computable embeddings. The class of undirected graphs are universal for effective interpretability. We give examples of graphs that are not Medvedev reducible to any linear ordering, or to the jump of any linear ordering, but any graph can be coded in a linear ordering using computable Σ_3 formulas. Friedman and Stanley [1] gave a Turing computable embedding L of directed graphs in linear orderings. We show that there do not exist $L_{\omega_1\omega}$-formulas that uniformly interpret the input graph G in the output linear ordering $L(G)$. This is joint work with J. Knight, and S. Vatev [3].

We have also a positive result—we prove that the class of fields are effectively interpreted in the class of Heisenberg groups, improving on and generalizing a 1960 result of Mal'tsev. The second part is a joint work with R. Alvir, W. Calvert, G. Goodman, V. Harizanov, J. Knight, R. Miller, A. Morozov, and R. Weisshaar.

References

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Generalized Computability in Approximation Spaces

We consider generalized hyperarithmetical computability and some of its applications. A series of positive results related to the problem stated by Yu.L. Ershov on the structure of Sigma-degrees of dense linear orders is obtained. In particular, we prove that interval models of temporal logic, as well as finite fragments of approximation spaces generated by interval Boolean algebras, are Sigma-definable (effectively interpretable) in hereditarily finite superstructures over dense linear orders. These results are used in the analysis of semantics of verbs in natural languages within the approach for formal semantics proposed by R. Montague.

A Comnputability Theory of Real Numbers

Xizhong Zheng

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Abstract

A real number x is called *computably approximable* (c.a. for short) if it is the limit of a computable sequence (x_s) of rational numbers, i.e. $x = \lim x_s$. The c.a. reals are also called Δ_2^0-reals. If additionally the convergence is effective, i.e., there is an effective error estimation in the sense that $|x - x_s| \leq 2^{-s}$ for all s, then the real number x is called *computable*. If the effective convergence condition is replaced by some weaker conditions, various classes of real numbers of weak computabilities can be introduced. For example, if the sequence is increasing, then the limiting number is *computably enumerable* (c.e.); if the convergence is *weakly effectively* in the sense that the sum of jumps $\sum |x_s - x_{s+1}|$ is finite, then the limiting number is *d.c.e.* (difference of c.e. real numbers); if the convergence is *divergence bounded* in the sense that the number of non-overlapping index-pairs (i, j) with $|x_i - x_j| > 2^{-s}$ is bounded by $f(s)$ for some computable function f, then the limiting number is called *divergence bounded computable* (d.b.c. for short).

This talk will give a brief survey of these weak computabilities of c.a. real numbers. In particular, you will see that these classes of real numbers have very nice mathematical properties as well as very natural computational properties. They are also related closely to the relative randomness of different levels.