A Computability Theory of Real Numbers

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Classical Computability Theory

The classical computability theory

- defines the computability and reducibility of sets and functions;
- is interested mainly in the non-computable objects;
- explores the levels of uncomputability (like unsolvability degrees) and the related structures;
- handles only the discrete structures of countable sets like $\mathbf{N}$ or Σ^*;
- is not able to deal with the real numbers and real functions.

However, the computation related to the real numbers is one of the most important tasks in practice.

A computability theory of real numbers is important for theoretical research as well as for applications (e.g. in model checking with hybrid systems — E.M.Clerke 2012 — "Turing's computable real numbers and why they are still important today.").

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Which Real Numbers Are Computable?

Two natural criteria:

- Good computational properties

  the computable real numbers should be somehow "calculable";

- Good mathematical properties

  e.g., the class of computable real numbers should be closed under arithmetical operations and computable functions.

What is a reasonable definition of computable real numbers?

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The First Attempt

CA $= \{$the limits of computable sequences of rational numbers$\}$ — class of c.a. reals.

The class **CA** has good mathematical properties:

- **CA** is closed under the arithmetical operations $+, -, \times$ and $\div$, i.e., it is a field.
- **CA** is closed under computable real functions.
- **CA** $= \Delta_2$, i.e., $x_A \in$ **CA** iff $A \in \Delta_2$, where $x_A := 0.A = \sum_{i \in A} 2^{-(i+1)}$.

CA does not have good computability theoretical property — not good enough!

A computable sequence (x_s) of rationals does not supply any "useful" information about its limit $x := \lim x_s$ in any finite moment.

E.g, after any finitely many steps

- we do not have an upper or lower bound of x;
- we cannot write down definitively any digital of the decimal expansion of x.

CA — minimal requirement of the computability of reals.

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Definition of Alan Turing (1936):

A real number is computable if its decimal expansion is calculable by finite means.

"finite means" $\Longrightarrow$ "automatic machine" (Turing machine)

Church-Turing thesis:

TM computability = intuitive computability

More precisely:

$x \in [0,1]$ is computable $\iff$ $x = 0.f(0)f(1)f(2)\ldots$ for a computable function f.

Some examples of computable real numbers (Turing 1936):

all rational numbers (e.g., $\frac{1}{3}$);

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Equivalent Definitions

Theorem of Raphael Robinson (1951): The followings are equivalent:

- (Decimal representation) x is computable;
- (Binary representation) $x = x_A := 0.A = \sum_{n\in A} 2^{-(n+1)}$ for a computable set $A \subseteq \mathbf{N}$;
- (Dedekind cut representation) $L_x := \{r \in \mathbf{Q} : r < x\}$ is a computable set;
- (Cauchy representation) There is a computable sequence (x_s) of rationals which converges to x effectively in the sense

$$(\forall n)(|x - x_n| \leq 2^{-n}) \quad \text{or} \quad (\forall n)(|x_n - x_{n+1}| \leq 2^{-n}).$$

  (x is "effectively computable", **EC** $:= \{x : x \text{ is computable}\}$.)

- (Nested interval representation) There is a computable sequence $((a_s, b_s))$ of rational intervals such that

$$(\forall s)(a_s < a_{s+1} < x < b_{s+1} < b_s) \ \& \ \lim_{s\to\infty}(b_s - a_s) = 0.$$

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Properties of Computable Real Numbers

- The definition of omputable real numbers is very robust;
- Computable real numbers are calculable. (exact computation);
- The class of computable real numbers is closed under the arithmetical operations;
- The class of computable real numbers is closed under computable operators (computable functions).
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Primitive Recursive Real Numbers

Specker (1949) defined primitive recursive reals in the following ways.

- PR_3 — by Dedekind's cuts
- PR_2 — by Decimal expansions
- PR_1 — by Cauchy sequences
- PR_0 — by Nested interval sequences

Specker 1949 and Skordev 2001 have shown that

$$\mathsf{PR}_3 \subsetneq \mathsf{PR}_2 \subsetneq \mathsf{PR}_1 \subsetneq \mathsf{PR}_0 = \mathsf{EC}$$

PR_1 is widely accepted as the definition of "primitive recursive reals" due to its good mathematical properties.

More complicated for the polynomial time computable real numbers.

Examples of Non-Computable Real Numbers

Example of Specker (1949):

A set A is c.e. if it has a computable enumeration — a computable sequence (A_s) of finite sets such that

$$A_0 = \emptyset, \qquad (\forall s)(A_s \subseteq A_{s+1}), \qquad \bigcup A_s = A.$$

The real number $x_A := \sum_{n \in A} 2^{-(n+1)}$ is not computable, if the set A is c.e. but not computable.

Remark:

The real number x_A is the limit of an increasing computable sequence(x_s) of rational numbers defined by $x_s := x_{A_s}$;

Consequence:

The limit of an increasing computable sequence of rational numbers is not necessarily computable.

Examples of Non-Computable Real Numbers

Example of Specker (1949):

A set A is c.e. if it has a computable enumeration — a computable sequence (A_s) of finite sets such that

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x is left computable if it is the limit of an increasing computable sequence (x_s) of rationals.

$x \in \mathbf{LC} \iff L_x := \{r \in \mathbb{Q} : r < x\}$ is a c.e. set.

(l.c. reals are also called c.e. or left-c.e.)

Theorem. [Soare 1969, Ambos-Spies et al 2000, Calude et al 2001] *x is l.c. iff $x = 0.A$ for a strongly ω-c.e. set A. Where a set A is strongly ω-c.e. if there is a computable sequence (A_s) of finite sets which convergences to A such that*

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x is right computable if $-x$ is l.c. (**RC**, or co-c.e.).

x is semi-computable if it is l.c. or r.c. ($\mathbf{SC} := \mathbf{LC} \cup \mathbf{RC}$).

Remark: x is s.c. iff there is a computable sequence (x_s) of rational numbers converging to x monotonically in the sense that $(\forall s, t)(s > t \Longrightarrow |x - x_s| \leq |x - x_t|)$.

Theorem. [Ambos-Spies, Weihrauch and Z. 2000]
If $A, B \subseteq \mathbf{N}$ are Turing incomparable c.e. sets, then the real number $x := x_{A \oplus \overline{B}}$ is not semi-computable.

Remark:

- $x_{A \oplus \overline{B}} = (x_{2A} + 1/3) - x_{2B+1}$.
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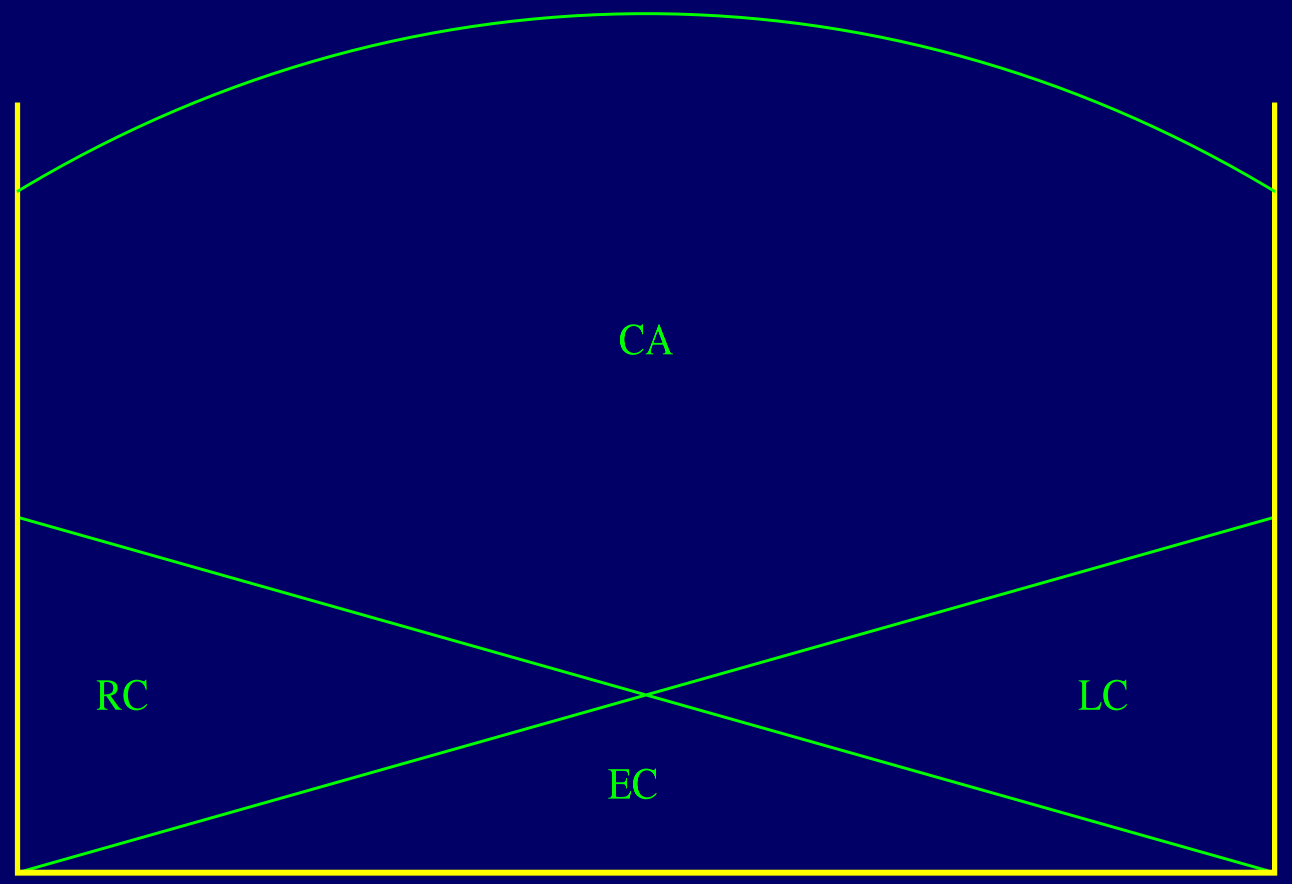

Example of Left Computable Reals

The length of a curve.

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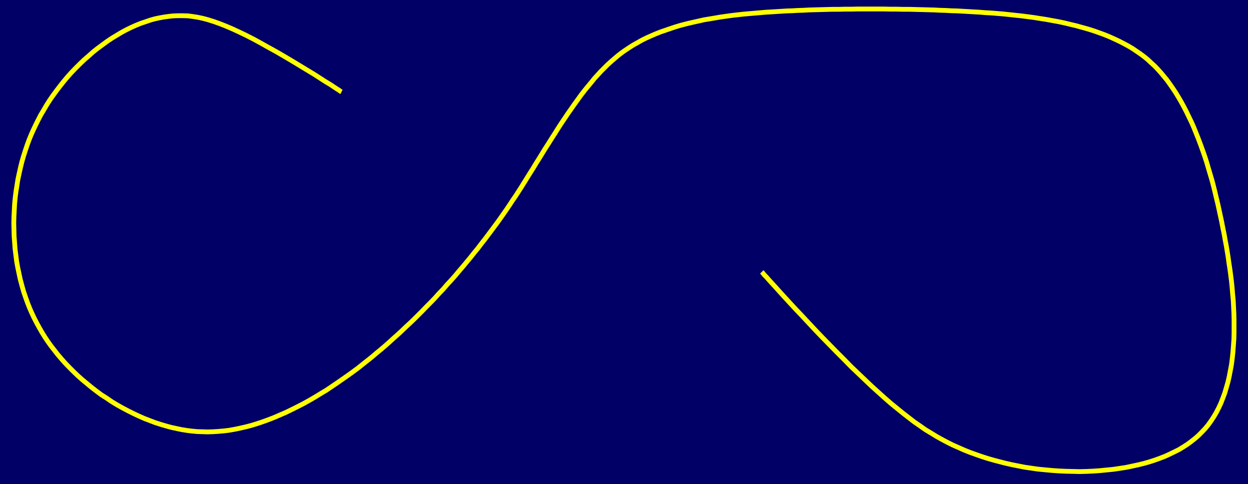

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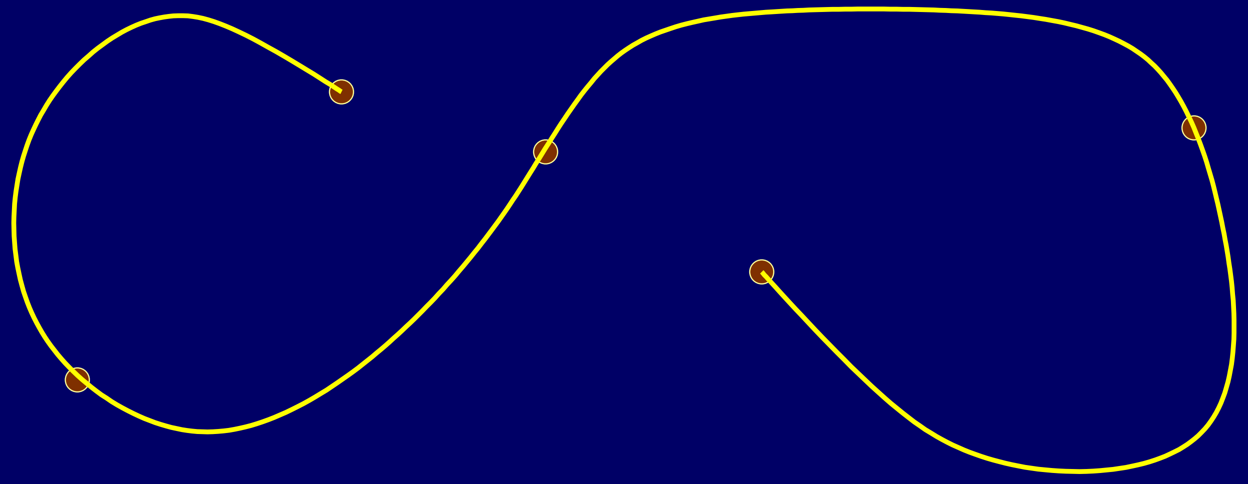

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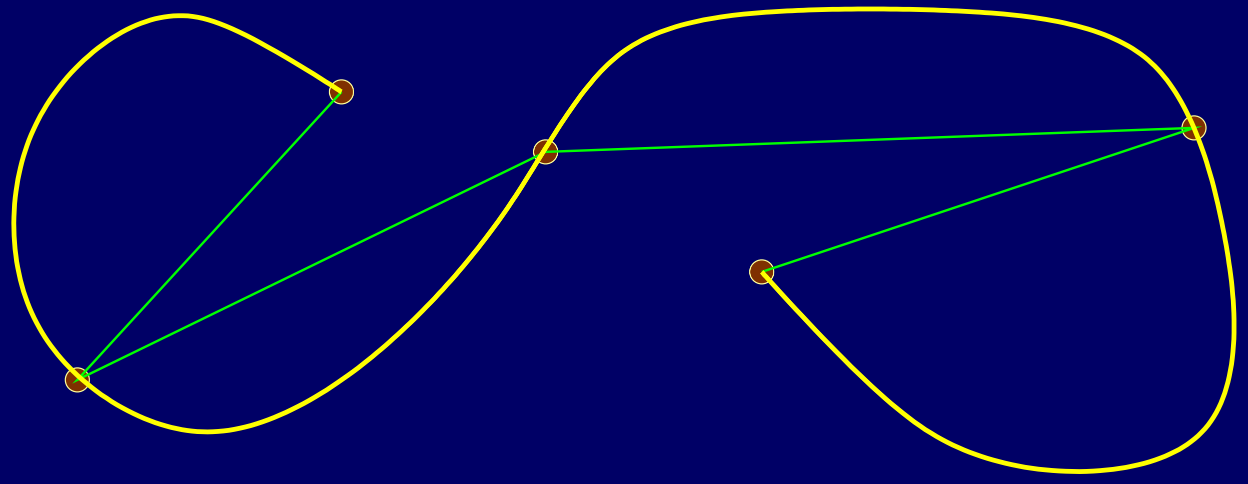

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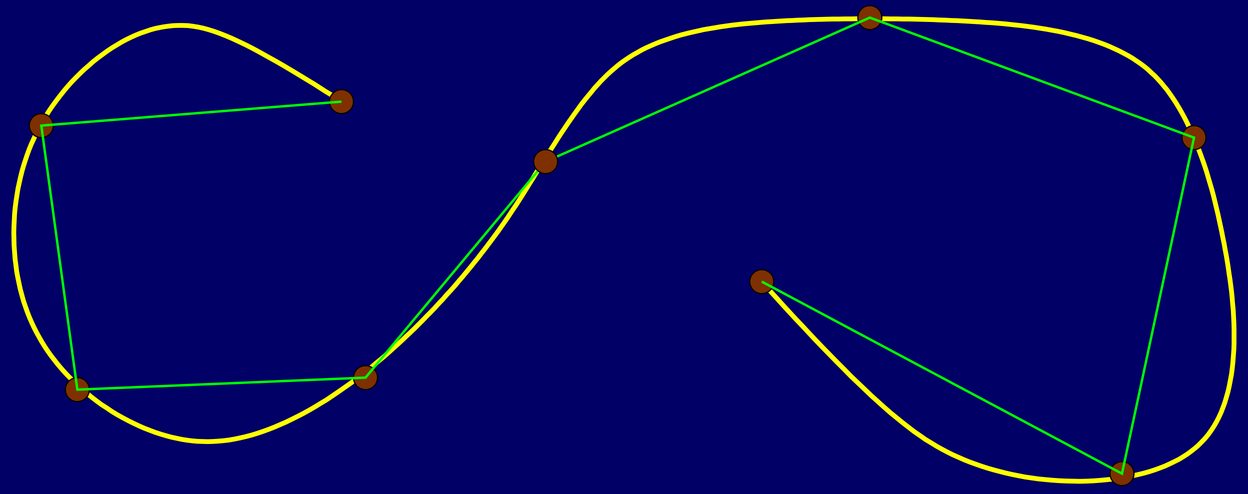

By increasing the cut points the polygon approximates the curve.

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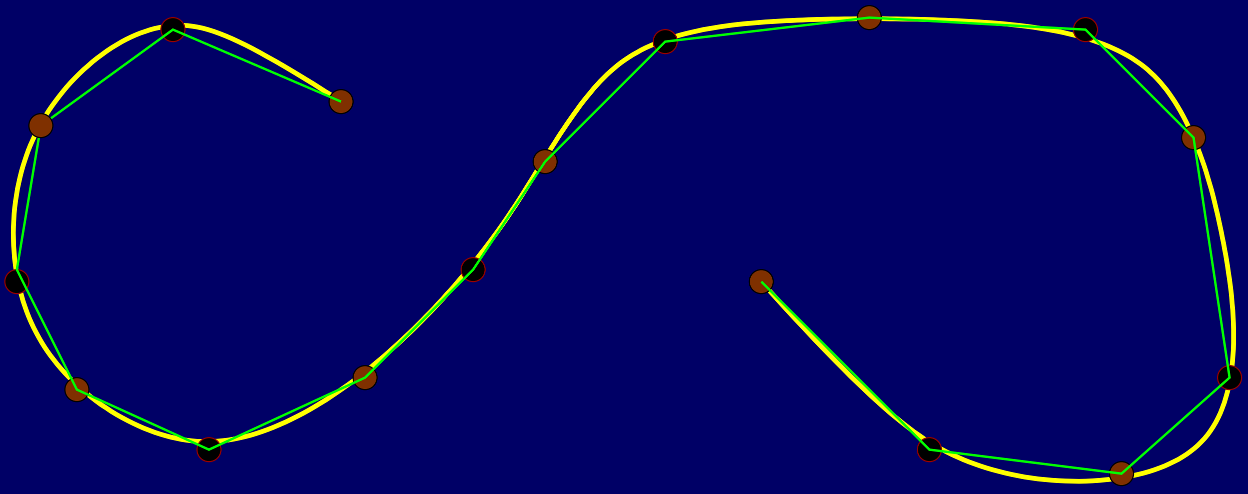

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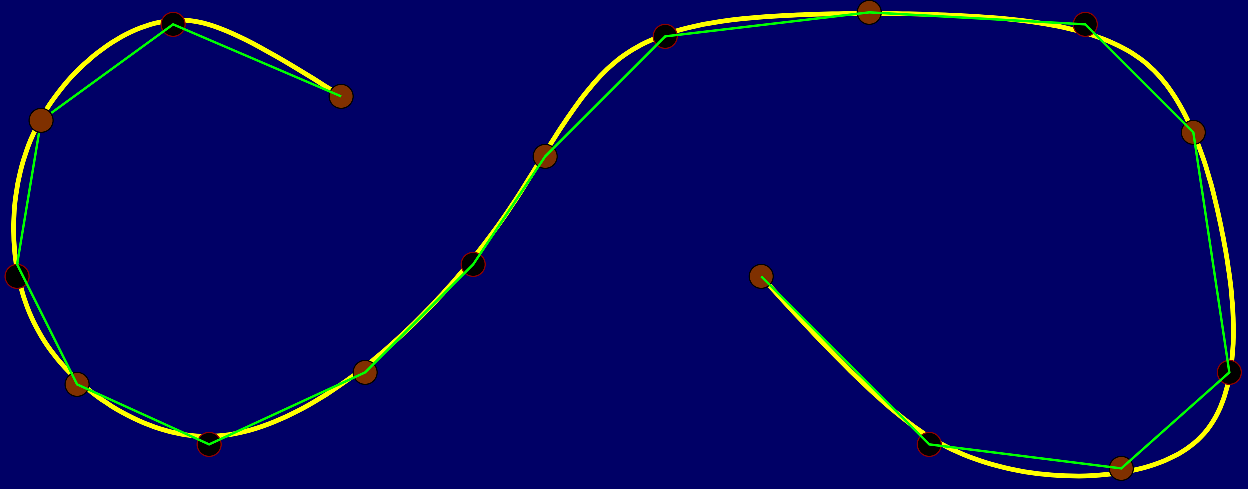

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The length of the curve is defined as the limit $\lim_{n\to\infty} l_n$, where l_n is the length of the polygon with $n+1$ cut points.

Remark: All lengths l_n are lower bounds of the length of the curve.

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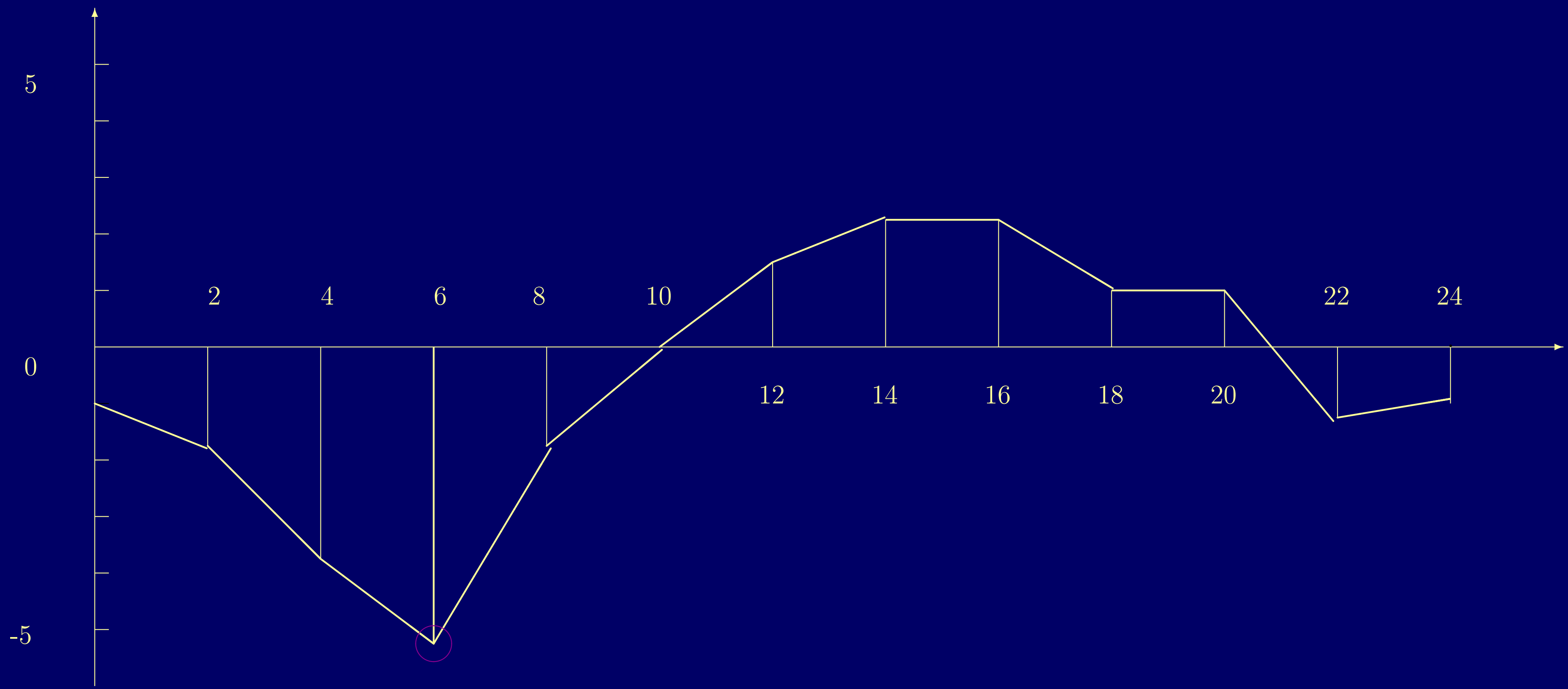
The minimal temperature of a day.



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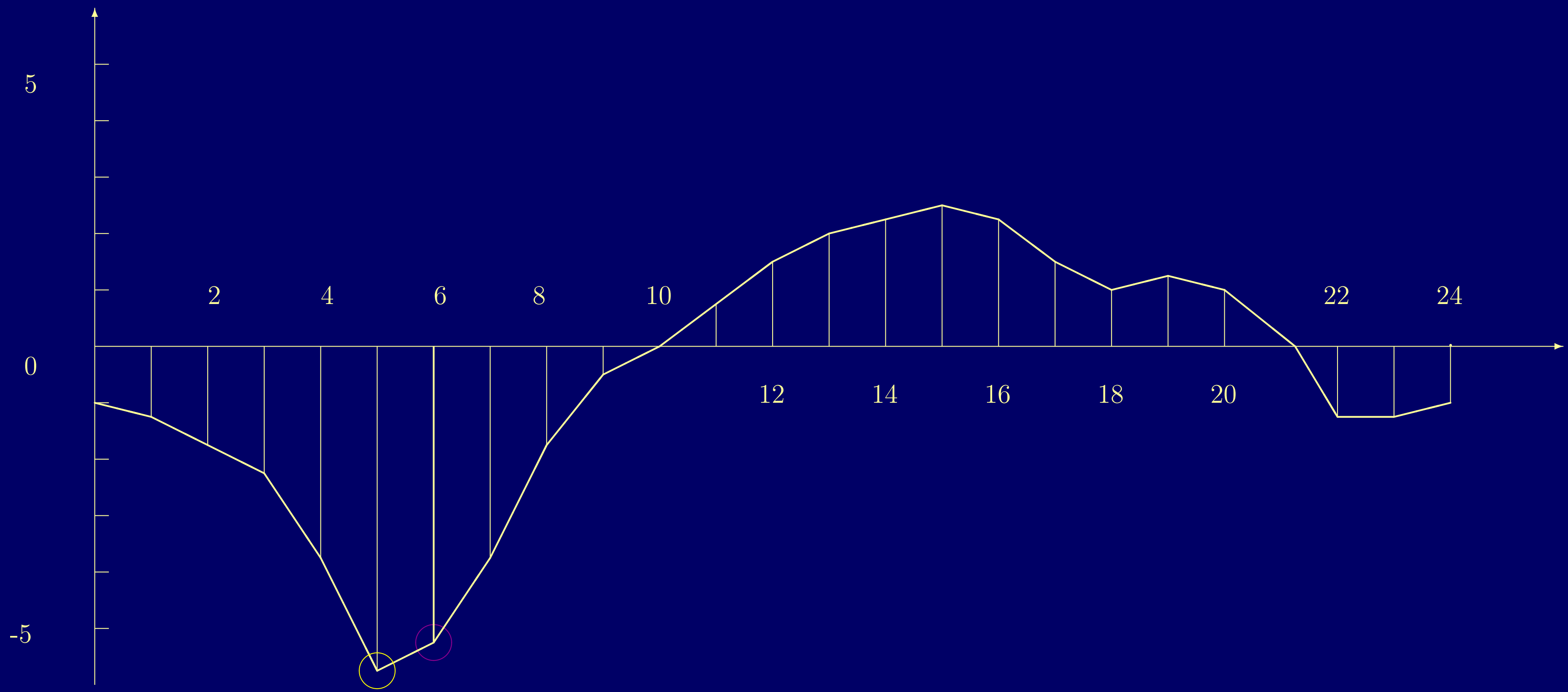

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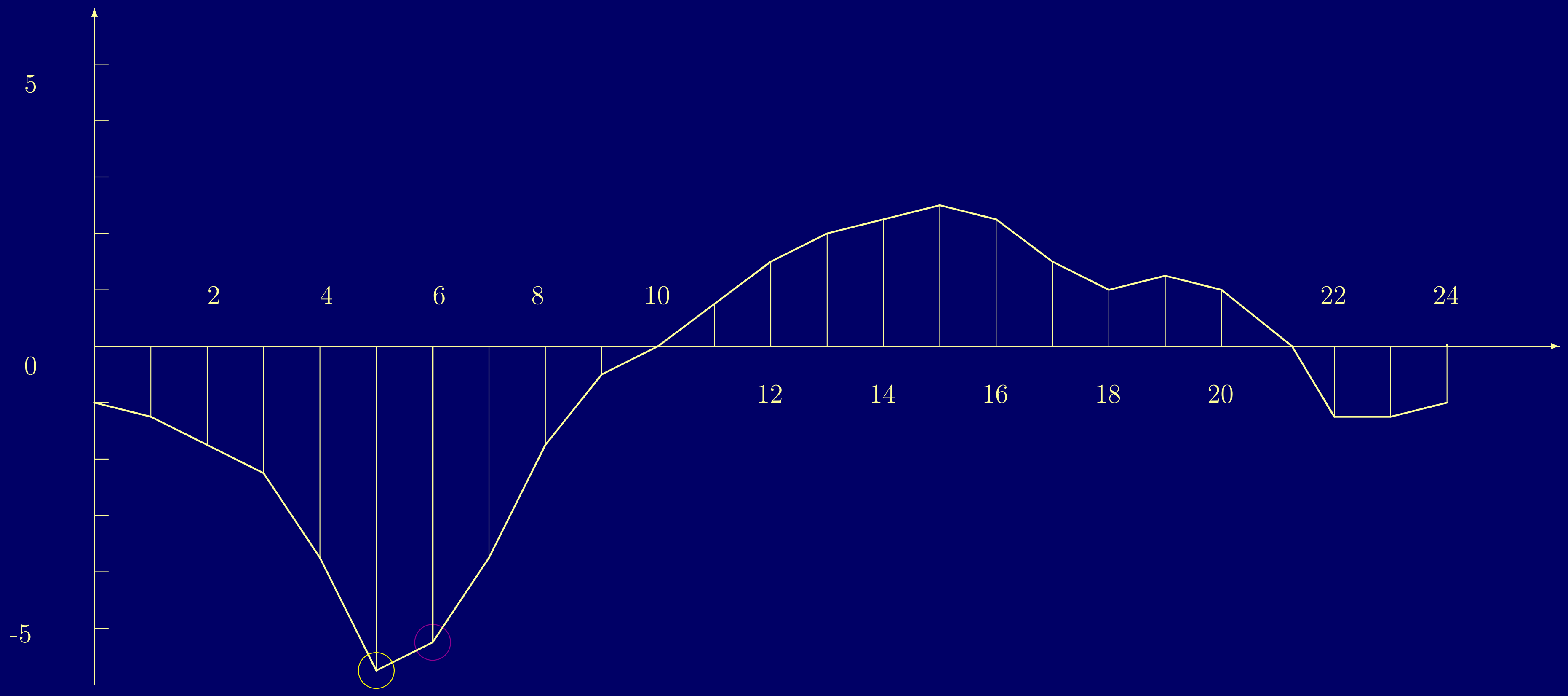

Problem: The class **SC** is not closed under the arithmetical operations!

Weakly Computable Reals (D-C.E.)

Definition. *A real x is called d-c.e. if $x = y - z$ for left computable reals y, z.*

The class **DCE** — difference of c.e.

Theorem. [Ambos-Spies, Weihrauch, Z. 2000] *x is d-c.e. iff there is a computable sequence (x_s) of rationals which converges weakly effectively to x in the sense that,*

$$\sum |x_s - x_{s+1}| \leq \infty.$$

Remark: (x_s) converges effectively if $|x_s - x_{s+1}| \leq 2^{-s}$ for all s. Then $\sum |x_s - x_{s+1}| \leq 2$

D-c.e. reals are also called weakly computable, (**WC** $=$ **DCE**)

Theorem. [AWZ2000, Ng2005 and Raichev2005]

- **WC** $= \mathrm{Arithm(SC)}$.
- **WC** *is a real closed field.*
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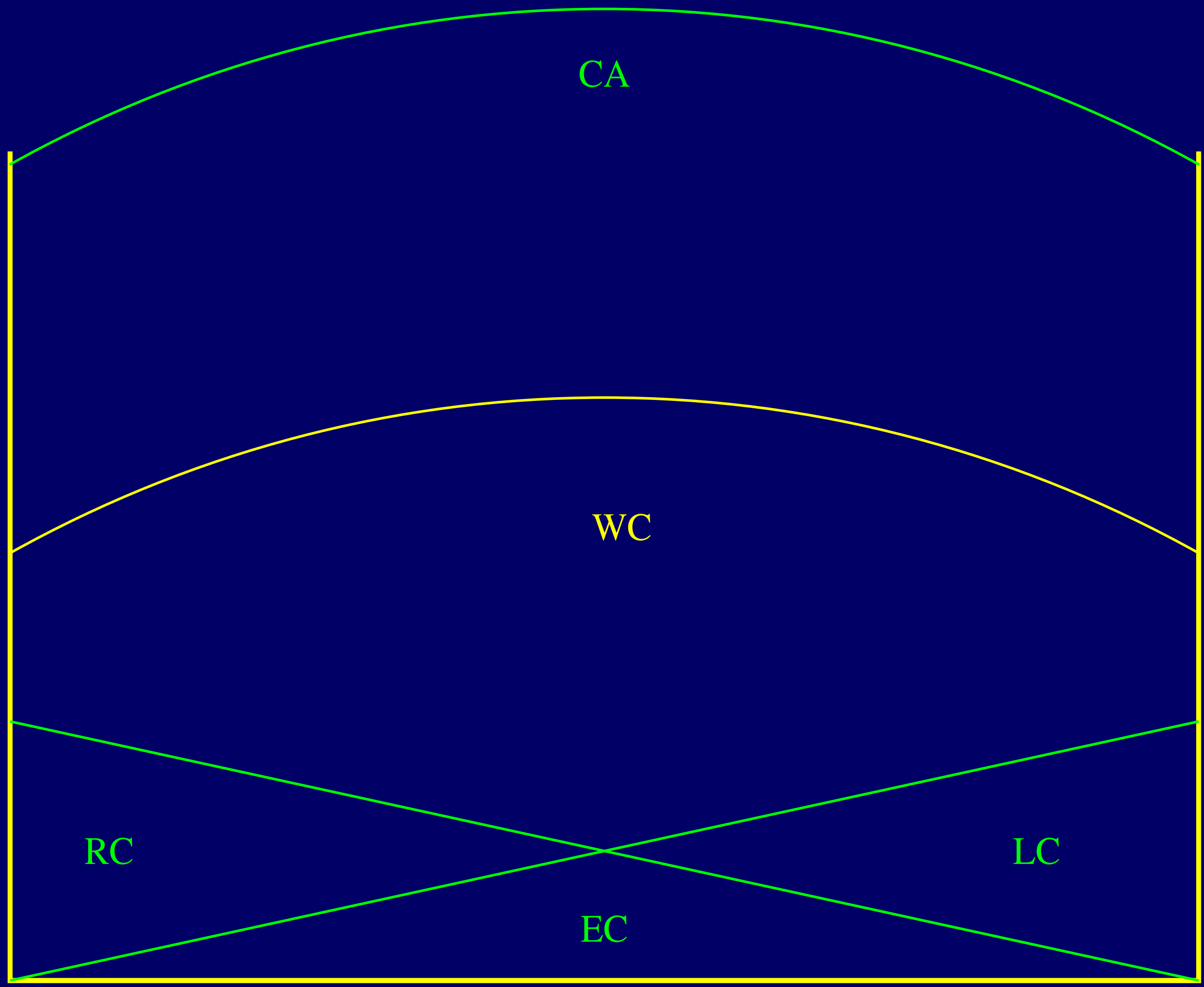

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A sequence (x_s) converges c.e. bounded if $(\forall s)(|x - x_s| \leq \sigma_s)$ where (σ_s) is a computable sequence of c.e. reals which converges to 0. ($\sigma_s := \sum_{i \geq s} \delta_i$ for a computable sequence (δ_s) of rationals such that the sum $\sum_s \delta_s$ is finite.)

Theorem. [Retting and Z. 2005] *A real number x is d-c.e. iff there is a computable sequence (x_s) of rational numbers which converges to x c.e. bounded.*

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The fifth characterization of **DCE** related to relative randomness.

The Fourth Characterization of D-c.e. Reals

A sequence (x_s) converges c.e. bounded if $(\forall s)(|x - x_s| \leq \sigma_s)$ where (σ_s) is a computable sequence of c.e. reals which converges to 0. ($\sigma_s := \sum_{i \geq s} \delta_i$ for a computable sequence (δ_s) of rationals such that the sum $\sum_s \delta_s$ is finite.)

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- The Kolmogorov complexity of a binary word σ relative to a Turing machine M is

$$K_M(\sigma) := \min\{|\tau| : M(\tau) = \sigma\}.$$

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$$(\exists c)(\forall n)(K(A \upharpoonright n) \geq n - c).$$

- A real number is called random if its binary expansion is a random sequence.

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Definition. *A function $f : \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitz if for each $\vec{x} \in \mathrm{dom}(f)$, there is a neighborhood U of $\vec{x}$ and a constant L such that*

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Corollary. *The class $S(\leq d)$ is a closed field for any c.a. reals d.*

Theorem. [Rettinger and Z. 2004] $S(\leq \Omega) = \mathbf{WC}$

Proof idea:

$S(\leq \Omega)$ contains all c.e. real and is a field $\Longrightarrow \mathbf{WC} \subseteq S(\leq \Omega)$;

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$$(\forall \vec{u}, \vec{v} \in U)(|f(\vec{u}) - f(\vec{v})| \leq L \cdot |\vec{u} - \vec{v}|)$$

Theorem. *Let d be a c.a. real. The class $S(\leq d) := \{y : y \leq_S^2 d\}$ is closed under locally Lipschitz computable functions.*

Corollary. *The class $S(\leq d)$ is a closed field for any c.a. reals d.*

Theorem. [Rettinger and Z. 2004] $S(\leq \Omega) = \mathbf{WC}$

Proof idea:

$S(\leq \Omega)$ contains all c.e. real and is a field $\Longrightarrow \mathbf{WC} \subseteq S(\leq \Omega)$;

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Solovay Completeness for DCE

Theorem. [Rettinger and Z. 2004]

1. *If d is a c.e. random real number, then $S(\leq d) = \mathbf{DCE}$;*

2. *The c.e. random reals are S-complete for* **DCE***;*

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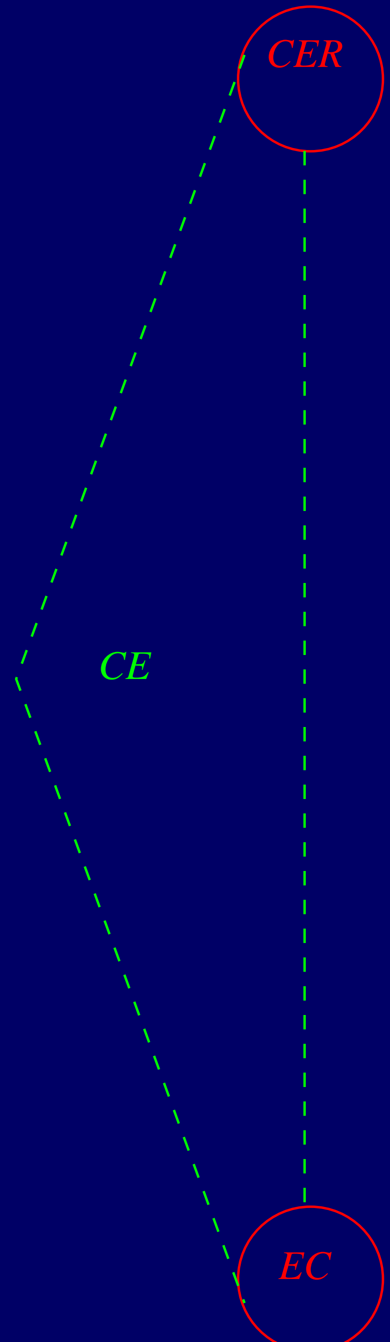
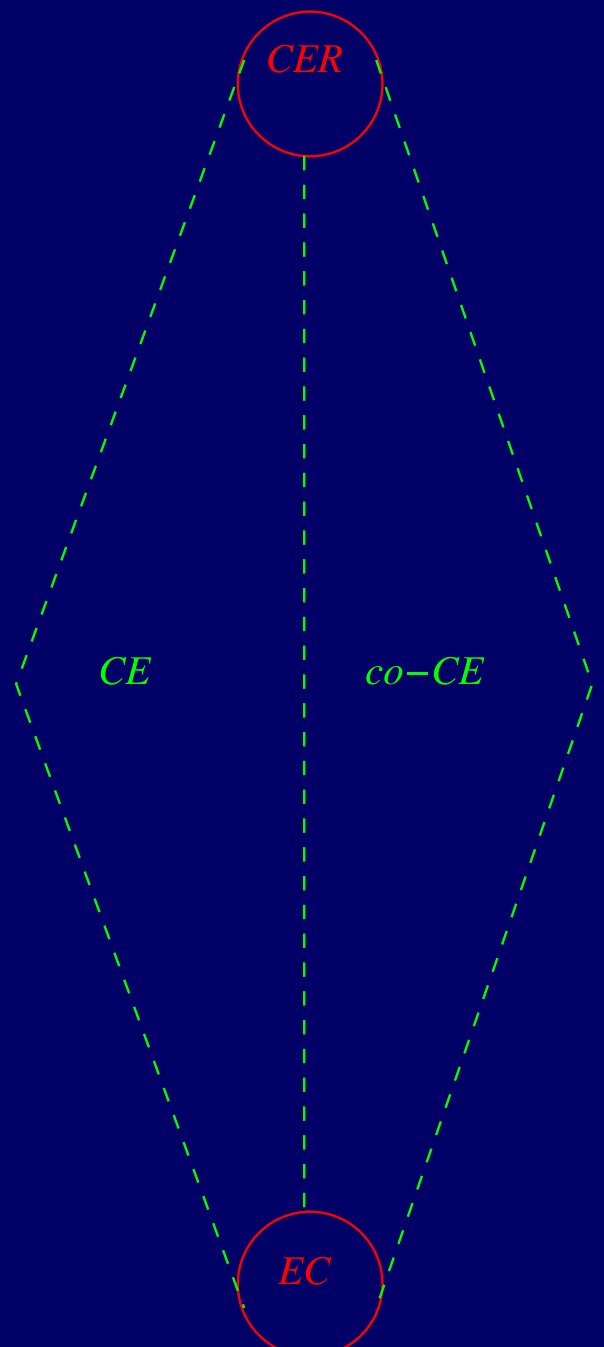
   (Co-c.e. reals are the limits of decreasing computable sequences of rationals.)

EC

CER

EC


CER
CE
EC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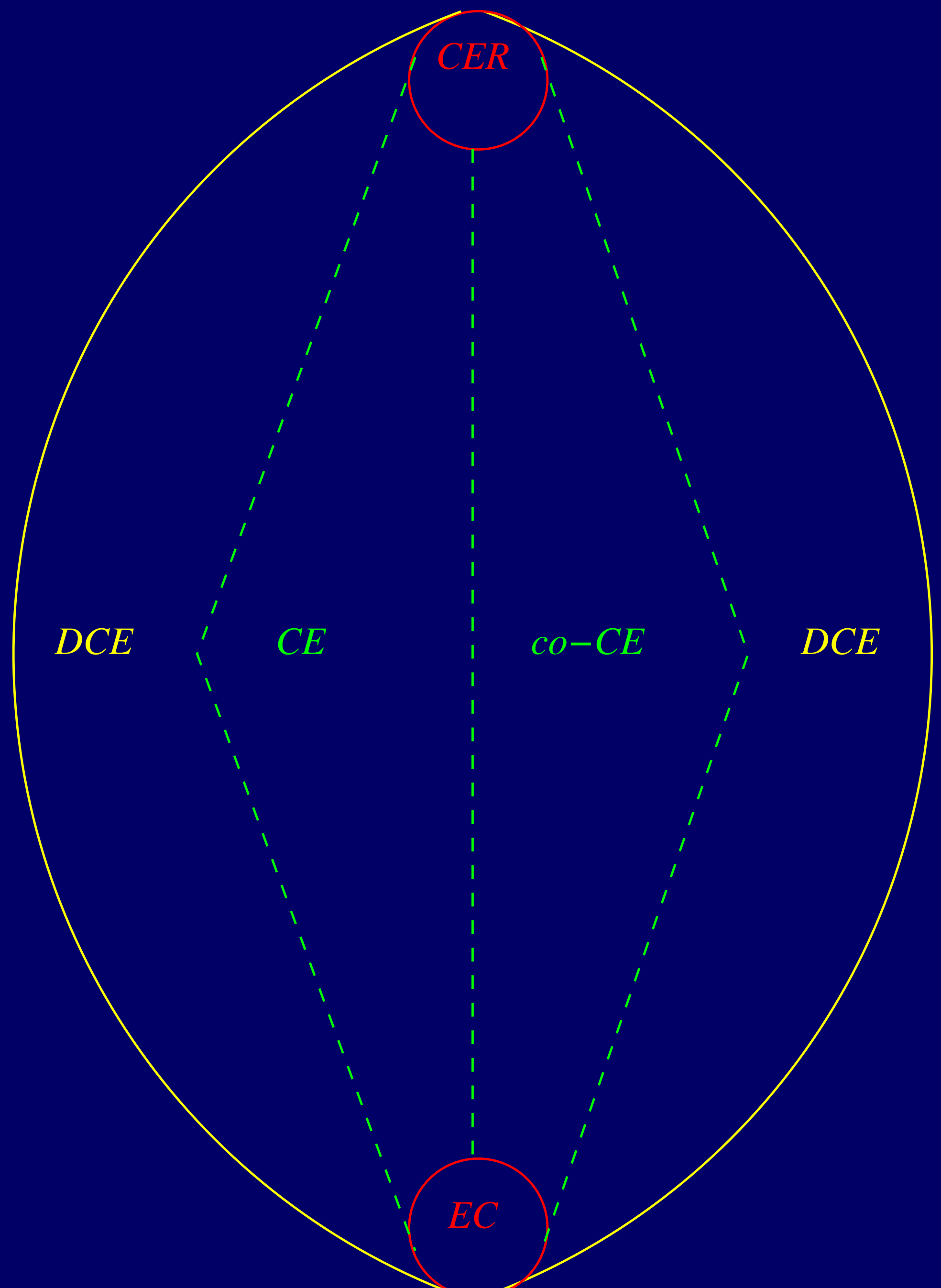
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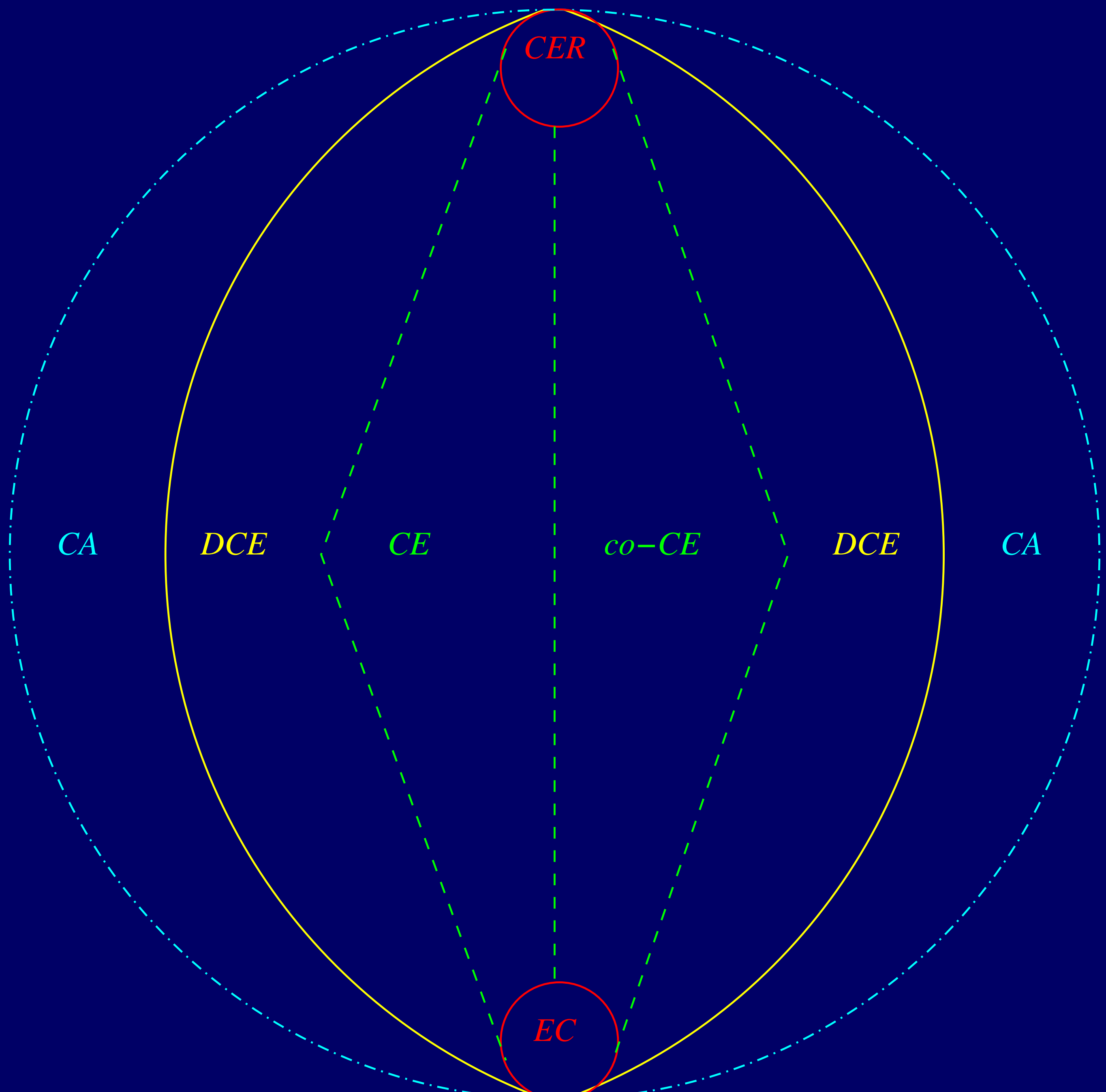
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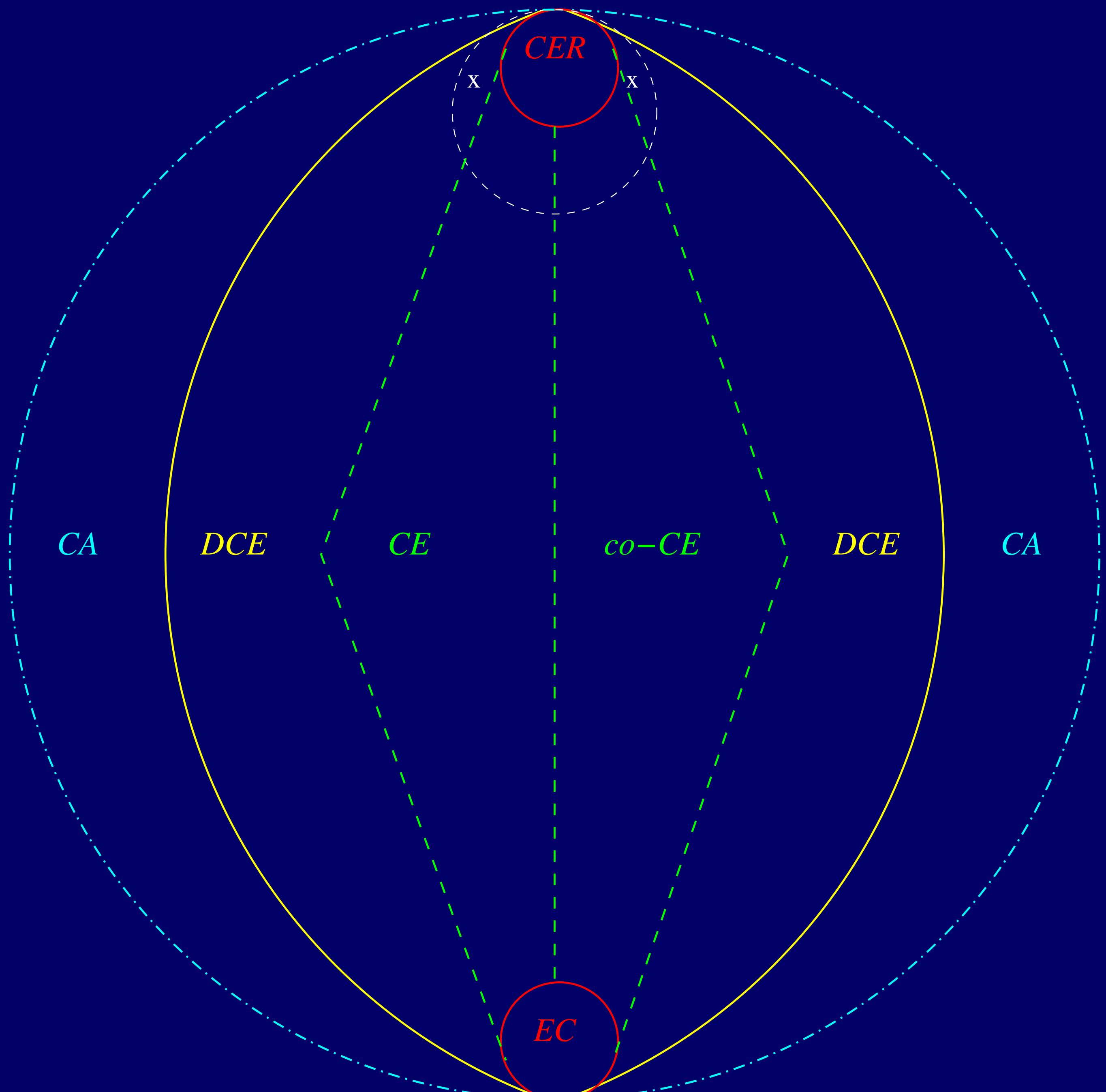
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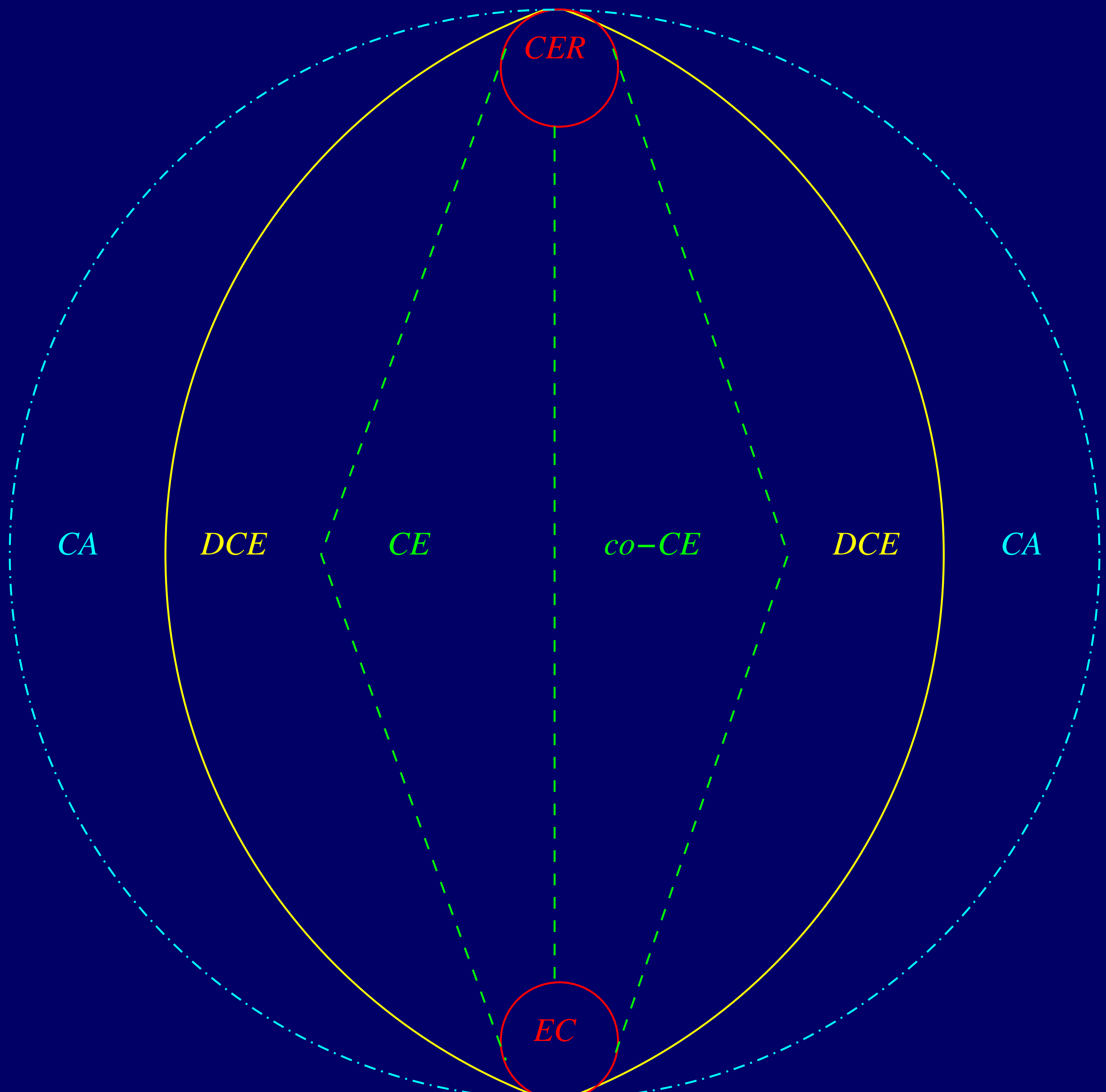
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Five Characterizations of DCE

The class **DCE** has at least five equivalent characterizations:

1. $x = y - z$ for $y, z \in$ **CE**
2. **DCE** $= \mathrm{Arithm}($**CE**$)$
3. Weakly computable.
4. C.e. bounded convergence
5. $x \leq_S^2 \Omega$.

Theorem. [Z. 2003, Downey, Wu, Z. 2004] *On the Turing degrees of d-c.e. reals, we have*

- *There is a d-c.e. real which does not have an ω-c.e. degree.*
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Derivation on DCE

Definition. [Miller 2017] *Let x be a d-c.e. and (x_s) be a computable sequence of rationals which converges to x weakly effectively. Let (Ω_s) be a computable increasing sequence of rationals which converges to Ω. Let*

$$\partial x = \lim_{s \to \infty} \frac{x - x_s}{\Omega - \Omega_s}$$

Theorem. [Miller 2017] *For any d-c.e. real x.*

- *∂x converges and is not dependent on the d-c.e. approximations of x.*
- *$\partial x = 0$ iff x is not random.*
- *$\partial x > 0$ iff x is a random left-c.e. real.*
- *$\partial x < 0$ iff x is a random right-c.e. real.*
- *The class of nonrandom d-c.e. reals forms a real closed field.*
- *If f is computable differentiable function and x is d-c.e. Then, $f(x)$ is d-c.e. and $\partial f(x) = f'(x)\partial x$ (DCE is closed under computable differentiable functions.)*

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The class DCE is not closed under total computable real functions.

Computable Real Functions

- Turing's promise (1936)
- Banach-Mazur (193?, 1963) — sequential computability
- Specker (1949) — (Primitive) recursive real functions — effective limits of (primitive) recursive sequences of (primitive) recursive functions on rational numbers.
- Grzegorczyk & Lacombe (1955) — sequential computability + effectively uniform continuity
- Weihrauch 1987 — Typ-2 Turing machine
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Closure under Computable Real Functions

The classes **EC** and **CA** are closed under the computable real functions.

Theorem. [Rettinger and Z. 2005] *The classes* **SC** *und* **WC** *are not closed under total computable real functions. But their closures are the same.*

Question: What is the closure of the classes **SC** and **WC** under total computable real functions?

Remark: The closure of real number classes under partial computable real functions is relative simple because of the following property of Ko:

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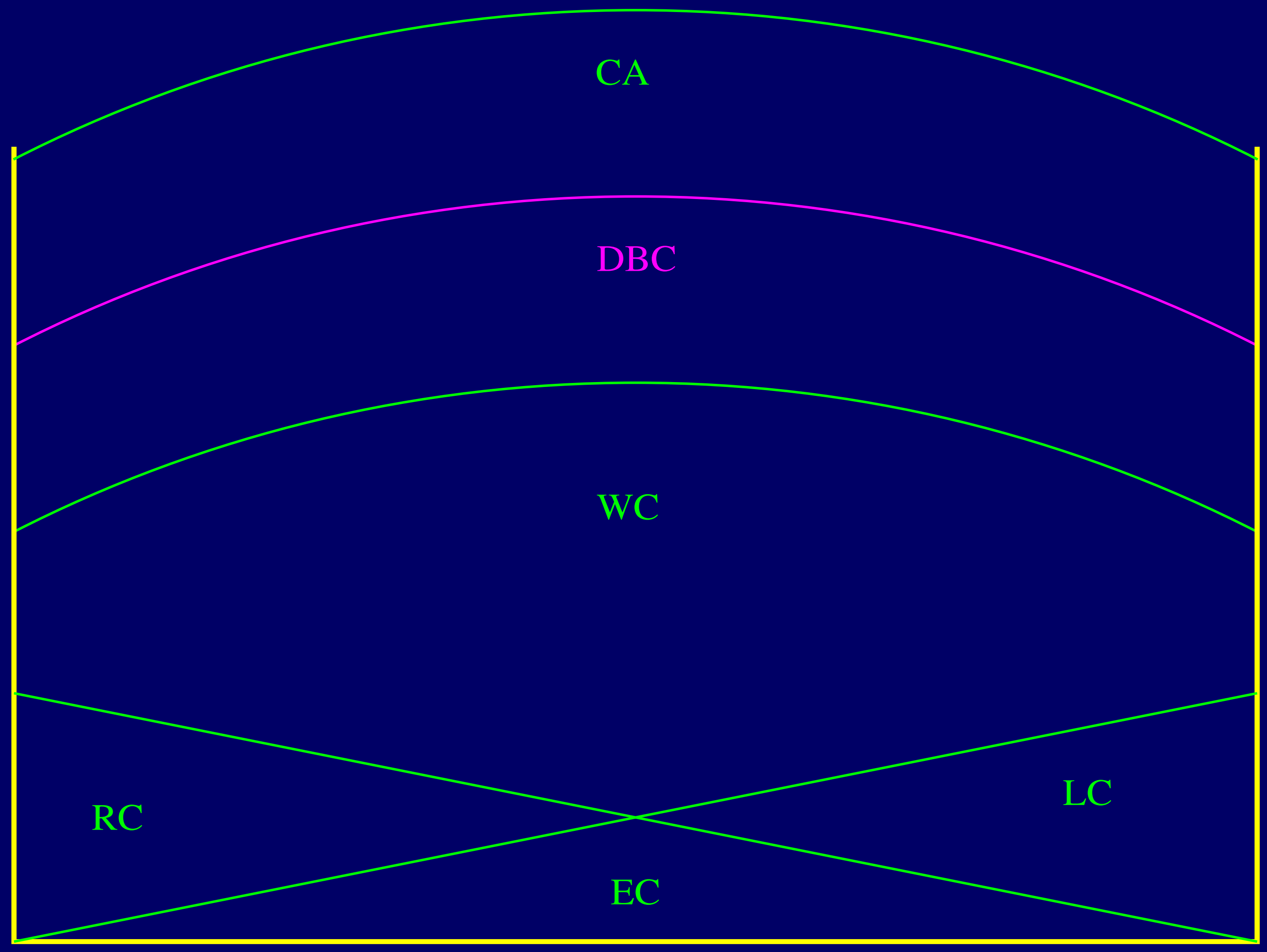

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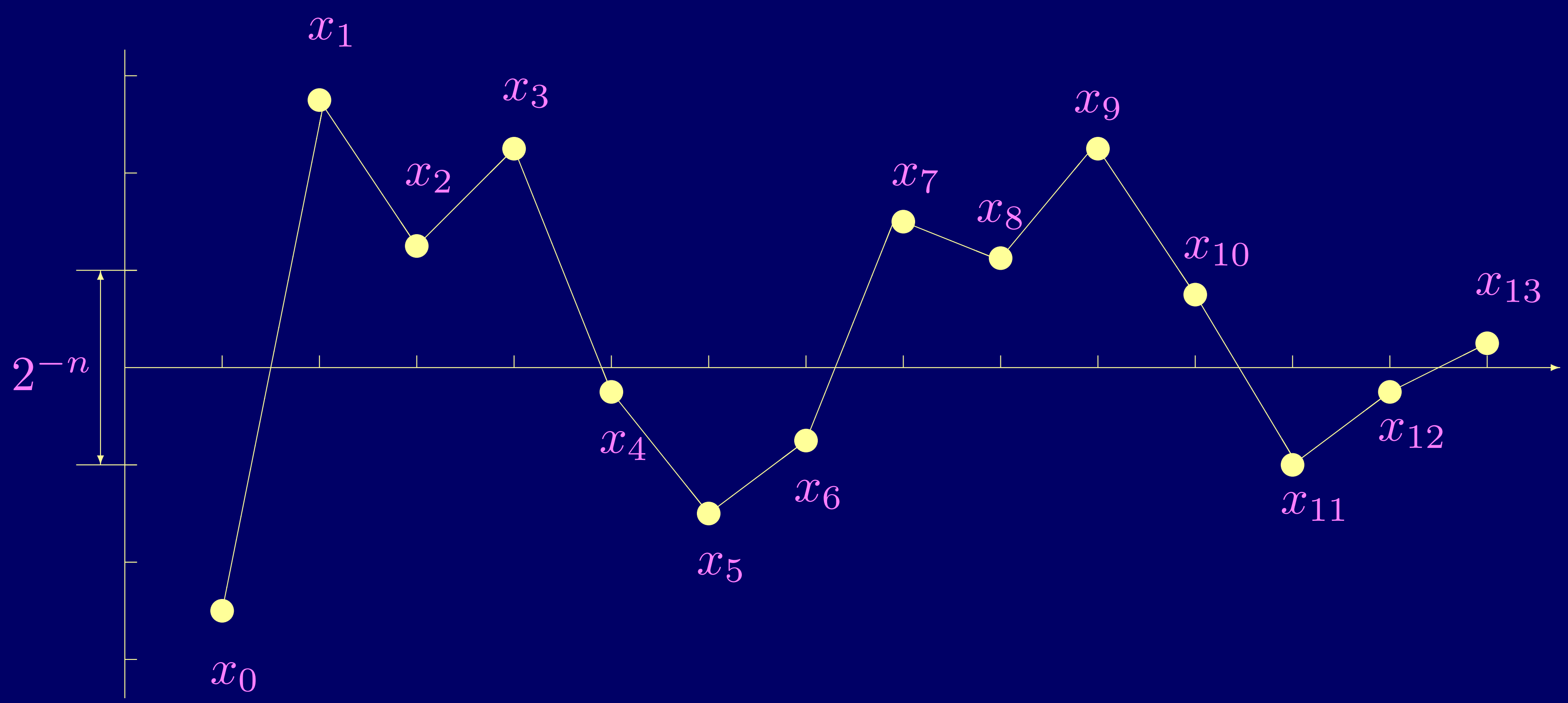

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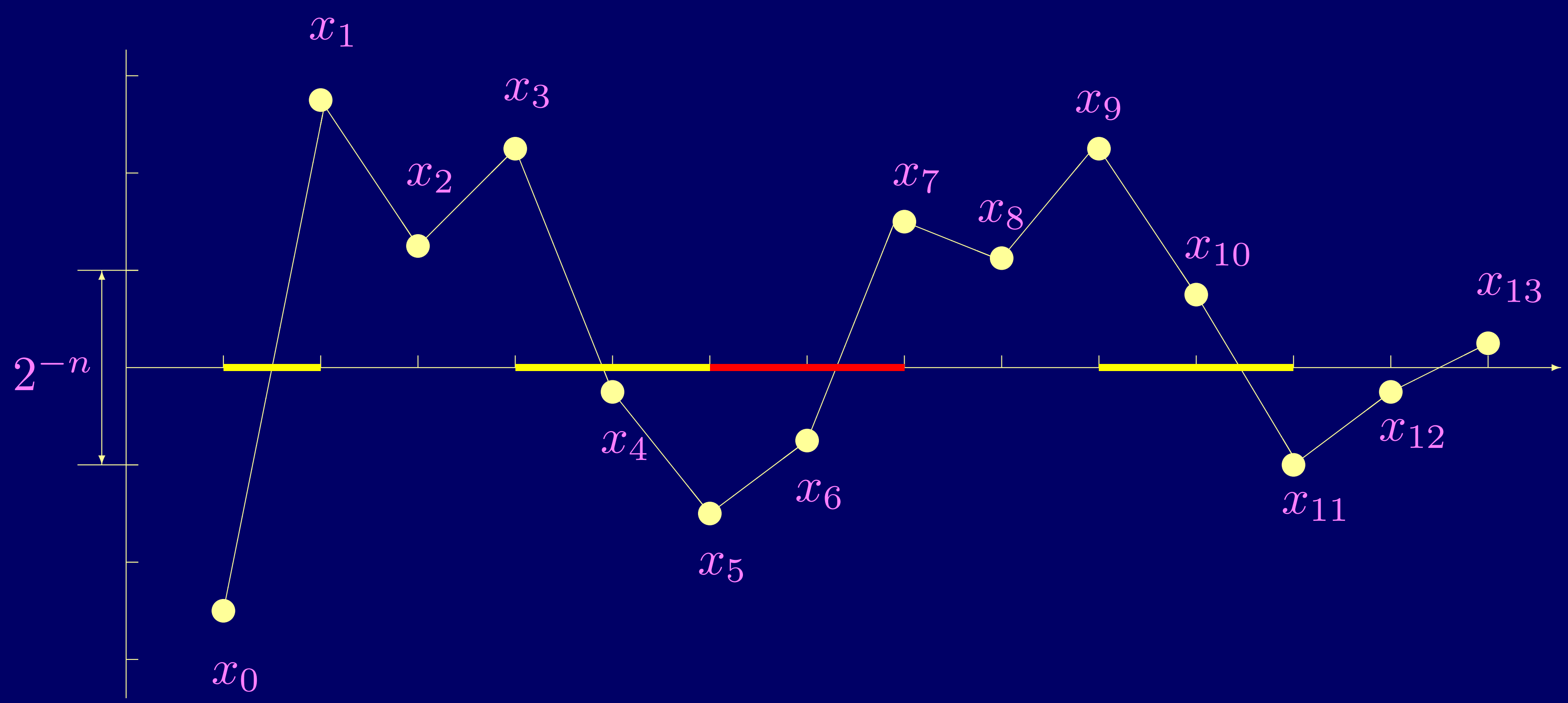

The index-pairs (0,1), (3,5), (5,7), (9,11) are four non-overlapping jumps of the size larger than 2^{-n}.

Divergence Bounded Convergence

Definition.

- *A sequence converges h-bounded, if it has at most $h(n)$ non-overlapping jumps of size larger than 2^{-n} for all n.*

- *A real x is called h-bc (bounded computable) if there is an h-bounded computable sequence of rationals which converges to x.*

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Theorem. [Rettinger and Z. 2005] *$x \in$ DBC iff there is a total computable function $h : \mathbf{N} \to \mathbf{N}$ and a computable sequence (x_s) of rationals which converges h-bounded to x. That is DBC $= C$-BC where C is the set of all total computable functions.*

DBC — divergence bounded computable

Theorem. [Rettinger and Z. 2005]

- DBC *is a field;*
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Theorem. [Rettinger and Z. 2005] $x \in$ DBC *iff there is a total computable function* $h : \mathbf{N} \to \mathbf{N}$ *and a computable sequence* (x_s) *of rationals which converges* h*-bounded to* x*. That is* DBC $= C$-BC *where* C *is the set of all total computable functions.*

DBC — divergence bounded computable

Theorem. [Rettinger and Z. 2005]

- DBC *is a field;*
- DBC *is strictly between the classes* DCE *and* CA*;*
- DBC *is the closure of the class* DCE *under total computable real functions.*

Convergence-Dominated Reducibility

Definition. [Rettinger and Z. 2018] *x is CD-reducible to y ($x \leq_{CD} y$) if there is a monotone total computable real function h with $h(0) = 0$ and two computable sequences (x_s) and (y_s) of rationals with $\lim x_s = x$, $\lim y_s = y$ and*

$$(\forall s)\left(|x - x_s| \leq h(|y - y_s|) + 2^{-s}\right)$$

(Extended Solovay: $(\forall s)\,(|x - x_s| \leq c(|y - y_s| + 2^{-s})))$

Lemma. *$x \leq_{CD} y$ iff there is a computable function $h : \mathbf{N} \to \mathbf{N}$ and two computable sequences (x_s) and (y_s) of rationals with $\lim x_s = x$, $\lim y_s = y$ and*

$$(\forall s, n)\left(|y - y_s| \leq 2^{-h(n)} \Longrightarrow |x - x_s| \leq 2^{-n} + 2^{-s}\right)$$

Theorem. [Rettinger and Z. 2018]

1. $x \leq_S^2 y \Longrightarrow x \leq_{CD} y$

2. $x \in$ DBC $\iff x \leq_{CD} \Omega$, i.e. DBC $= \mathrm{DC}(\leq_{DC} \Omega)$

Equivalent Characterizations of DBC

The class of DBC can be equivalently characterized in the following ways.

- Computable closure of DBC
- Computable closure of CE
- Class of d.b.c. reals ($\mathbf{DBC} = C$-$\mathbf{BC}$ for computable function class C)
- Class of reals which are CD-reducible to Ω.

Regarding the Turing degrees, we have

Theorem. [Rettinger and Z. 2005]

- *There exists Δ_2^0 degree which has no d.b.c real numbers.*
- *There exists d.b.c. degree which has no d-c.e. real numbers.*

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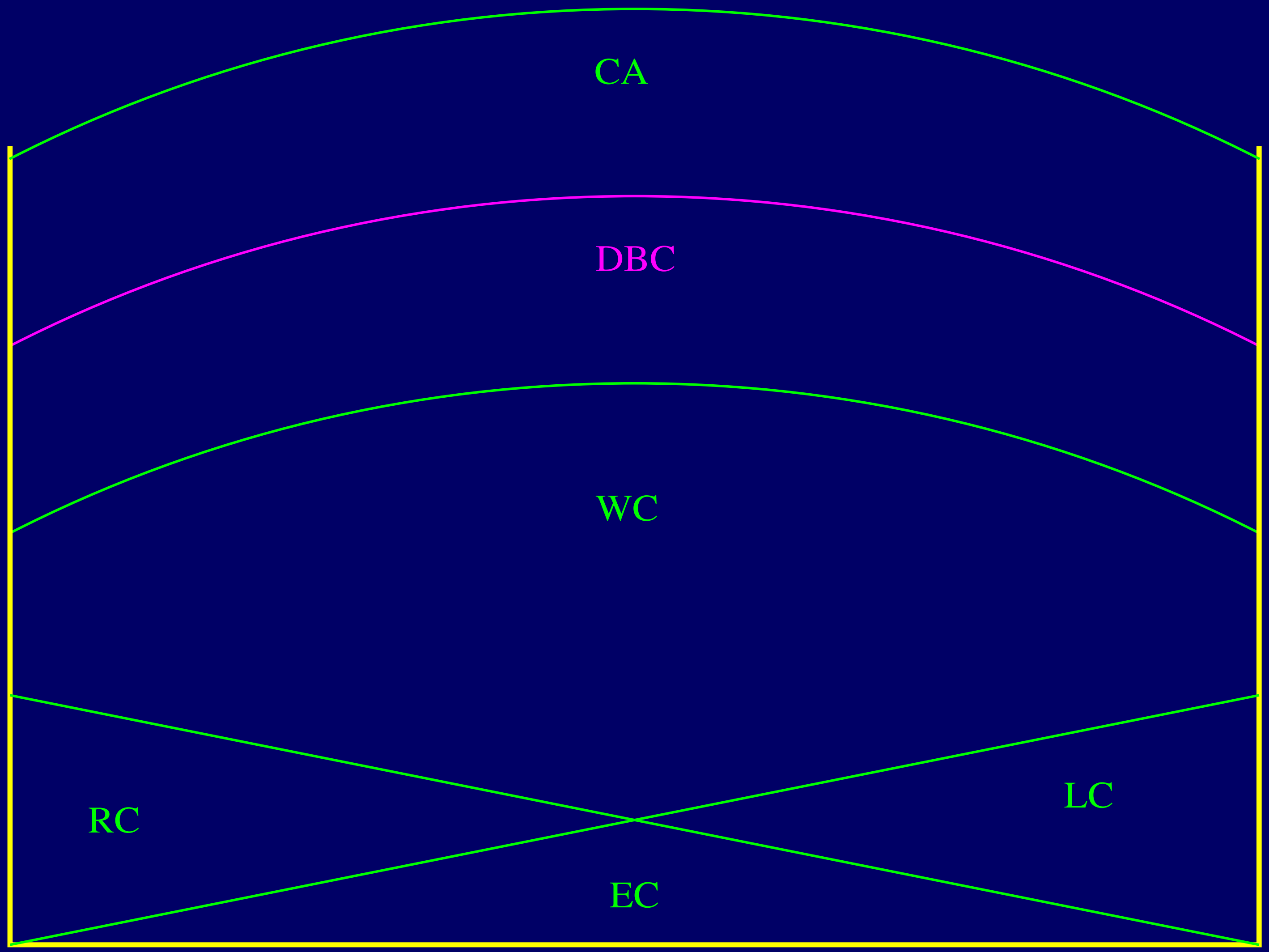
A Finite Hierarchy



Conclusion

1. Computability theory of real numbers is a subarea of a more comprehensive research area CCA (Computability and Complexity in Analysis). (http://cca-net.de/.)

2. The following classes of real numbers are explored in this talk:

3. There are further, also infinite, hierarchies of the class **CA**:

- the Ershov type hierarchies;
- h-monotone computability $(m > n \Longrightarrow |x - x_m| \le h(n)|x - x_n|)$.
- Turing degree hierarchies.
- etc.

4. The class **CA** is the second level (Δ_2^0) of the arithmetical hierarchy of real numbers.

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