# Generalized Computability in Approximation Spaces

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# Outline

- Effective Model Theory and Generalized Computability
- Approximation Spaces
  and Generalized Hyperarithmetical Computability
- Applications in Temporal Logic and Linguistics

# $\mathbb{HF}(\mathfrak{M})$

For a set $M$, consider the set $\mathrm{HF}(M)$ of hereditarily finite sets over $M$ defined as follows: $\mathrm{HF}(M) = \bigcup\limits_{n\in\omega} \mathrm{HF}_n(M)$, where

$\mathrm{HF}_0(M) = \{\varnothing\} \cup M$,
$\mathrm{HF}_{n+1}(M) = \mathrm{HF}_n(M) \cup \{a \mid a \text{ is a finite subset of } \mathrm{HF}_n(M)\}$.

For a structure $\mathfrak{M} = \langle M, \sigma^{\mathfrak{M}}\rangle$ of (finite or computable) signature $\sigma$, **hereditarily finite superstructure**

$$\mathbb{HF}(\mathfrak{M}) = \langle \mathrm{HF}(M); \sigma^{\mathfrak{M}}, U, \in, \varnothing\rangle$$

is a structure of signature $\sigma'$ (with $\mathbb{HF}(\mathfrak{M}) \models U(a) \iff a \in M$).

Remark: in the case of infinite signature, we assume that $\sigma'$ contains an additional relation $Sat(x, y)$ for atomic formulas under some fixed Gödel numbering.

**Fact**

$\mathbb{HF}(\mathfrak{M})$ *is the least admissible set over* $\mathfrak{M}$.

# $\Delta_0$-formulas and $\Sigma$-formulas

Let $\sigma' = \sigma \cup \{U^1, \in^2, \varnothing\}$ where $\sigma$ is a finite signature.

**Definition**

*The class of $\Delta_0$-**formulas** of signature $\sigma'$ is the least one of formulas containing all atomic formulas of signature $\sigma'$ and closed under $\wedge, \vee, \neg$, $\exists x \in y$ and $\forall x \in y$.*

**Definition**

*The class of $\Sigma$-**formulas** of signature $\sigma'$ is the least one of formulas containing all $\Delta_0$-formulas of signature $\sigma'$ and closed under $\wedge, \vee$, $\exists x \in y$, $\forall x \in y$ and $\exists x$.*

Let $\mathfrak{M}$ be a structure of a relational signature $\langle P_0^{n_0}, \ldots, P_k^{n_k} \rangle$ and let $\mathbb{A}$ be an admissible set.

**Definition (Yu. L. Ershov 1985)**

*$\mathfrak{M}$ is called* **$\Sigma$-definable in $\mathbb{A}$** *if there exist $\Sigma$-formulas* $\varphi(x_0, y), \psi(x_0, x_1, y), \psi^*(x_0, x_1, y)$, $\varphi_0(x_0, \ldots, x_{n_0-1}, y), \varphi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \varphi_k(x_0, \ldots, x_{n_k-1}, y)$, $\varphi_k^*(x_0, \ldots, x_{n_k-1}, y)$ *such that, for some parameter $a \in A$,* $M_0 \leftrightharpoons \varphi^{\mathbb{A}}(x_0, a) \neq \varnothing$, $\eta \leftrightharpoons \psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$ *is a congruence on* $\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0} \rangle$, *where*
$P_k^{\mathfrak{M}_0} \leftrightharpoons \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}$, $k \in \omega$,

$$\psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \psi^{\mathbb{A}}(x_0, x_1, a),$$

$\varphi_i^{*\mathbb{A}}(x_0, \ldots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \varphi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1})$ *for all $i \leqslant k$, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure* $\mathfrak{M}_0 / \eta$.

# $\Sigma$-definability of structures in admissible sets

$\Sigma$-definability of a model in an admissible set $\mathbb{A}$ is an extension (on computability in $\mathbb{A}$) of the notion of constructivizability of a model (in classical computability theory CCT).

For a countable structure $\mathfrak{M}$, the following are equivalent:

- $\mathfrak{M}$ is constructivizable (computable);
- $\mathfrak{M}$ is $\Sigma$-definable in $\mathbb{HF}(\varnothing)$.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_{\Sigma} \mathfrak{N}$ the fact that $\mathfrak{M}$ is $\Sigma$-definable in $\mathbb{HF}(\mathfrak{N})$.

For arbitrary cardinal $\alpha$, let $\mathcal{K}_\alpha$ be the class of all structures (of computable signatures) of cardinality $\leqslant \alpha$. We define on $\mathcal{K}_\alpha$ an equivalence relation $\equiv_\Sigma$ as follows: for $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$,

$$\mathfrak{M} \equiv_\Sigma \mathfrak{N} \text{ if } \mathfrak{M} \leqslant_\Sigma \mathfrak{N} \text{ and } \mathfrak{N} \leqslant_\Sigma \mathfrak{M}.$$

Structure

$$\mathcal{S}_\Sigma(\alpha) = \langle \mathcal{K}_\alpha / \equiv_\Sigma, \leqslant_\Sigma \rangle$$

is an upper semilattice with the least element, and, for any $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$,

$$[\mathfrak{M}]_\Sigma \vee [\mathfrak{N}]_\Sigma = [(\mathfrak{M}, \mathfrak{N})]_\Sigma,$$

where $(\mathfrak{M}, \mathfrak{N})$ denotes the model-theoretic pair of $\mathfrak{M}$ and $\mathfrak{N}$.

It is well-known that

$$\mathbb{C} \leqslant_\Sigma \mathbb{R}.$$

**Theorem (Yu. L. Ershov 1985)**

$$\mathbb{C} \leqslant_\Sigma \mathbb{L}$$

*for any dense linear order of size continuum.*

Motivation: find structures $\mathfrak{M}$ such that

1. $\mathfrak{M} \leqslant_\Sigma \mathbb{L}$ with $\mathbb{L}$ used essentially;
2. $\mathfrak{M}$ is "simple" yet natural and useful in applications.

Possible applications appear when $\mathbb{L}$ is treated as the scale of time.

**Definition (Yu. L. Ershov)**

1. *A first-order theory $T$ is called* **regular** *if it is decidable and model complete.*
2. *A first-order theory $T$ is called c-***simple** *(constructively simple) if it is decidable, model complete, $\omega$-categorical, and has a decidable set of the complete formulas.*

**Conjecture (Yu.L. Ershov, 1998)**

*Suppose a theory $T$ has an uncountable model which is $\Sigma$-definable in $\mathbb{HF}(\mathfrak{M})$, for some structure $\mathfrak{M}$ with a c-simple theory. Then $T$ has an uncountable model which is $\Sigma$-definable in $\mathbb{HF}(\mathbb{L})$ for some $\mathbb{L} \models \mathrm{DLO}$.*

The formal consequence of this conjecture is

**Conjecture**

*Any c-simple theory has an uncountable model which is $\Sigma$-definable in $\mathbb{HF}(\mathbb{L})$ for some $\mathbb{L} \models \mathrm{DLO}$.*

**Definition (S.)**

*A first-order theory $T$ is called sc-**simple** if it is decidable, submodel complete, $\omega$-categorical, and has a decidable set of the complete formulas.*

**Theorem (S. 2010)**

*Let $T$ be a sc-simple theory of finite signature. Then there exists an uncountable model $\mathfrak{M}$ of $T$ such that $\mathfrak{M}$ is $\Sigma$-definable in $\mathbb{HF}(\mathbb{L})$, $\mathbb{L} \models \mathrm{DLO}$.*

**Definition**

Structure $\mathfrak{A}$ is called $s\Sigma$-**definable** in $\mathbb{HF}(\mathfrak{B})$ (denoted as $\mathfrak{A} \leqslant_{s\Sigma} \mathfrak{B}$) if $A \subseteq \mathrm{HF}(B)$ is a $\Sigma$-subset of $\mathbb{HF}(\mathfrak{B})$, and all the signature relations and functions of $\mathfrak{A}$ are $\Delta$-definable in $\mathbb{HF}(\mathfrak{B})$.

### Theorem (Friedberg 1957)

*Let $A \subseteq \omega$ be a set such that $\mathbf{0}' \leqslant_T A$. There exists a set $B \subseteq \omega$ such that*

$$B' \equiv_T A.$$

### Theorem (A.Soskova, I.Soskov 2009)

*Let $\mathfrak{A}$ be a countable structure such that $\mathbf{0}' \leqslant_w \mathfrak{A}$. There exists a structure $\mathfrak{B}$ such that*

$$\mathfrak{B}' \equiv_w \mathfrak{A}.$$

### Theorem (S. 2009)

*Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leqslant_{s\Sigma} \mathfrak{A}$. There exists a structure $\mathfrak{B}$ such that*

$$\mathfrak{B}' \equiv_{s\Sigma} \mathfrak{A},$$

*where* $\mathfrak{B}' = (\mathbb{HF}(\mathfrak{B}), \Sigma\mathrm{-Sat}_{\mathbb{HF}(\mathfrak{B})})$.

## Definition (S. 2013)

*A structure* $\mathfrak{M}$ *is called* **quasiregular** *if*

$$\mathfrak{M}_{\mathrm{Morley}} \equiv_{s\Sigma} \mathfrak{M},$$

*where* $\mathfrak{M}_{\mathrm{Morley}}$ *is the Morley expansion of* $\mathfrak{M}$.

Let $\mathfrak{M}$ be a structure of signature $\sigma$, signature $\sigma_*$ consists of all symbols from $\sigma$ and function symbols $f_\varphi(x_1, \ldots, x_n)$ for all $\exists$-formulas $\varphi(x_0, x_1, \ldots, x_n) \in F_\sigma$. A structure $\mathfrak{M}_S$ of signature $\sigma_*$ is called **existential Skolem expansion of** $\mathfrak{M}$ if $|\mathfrak{M}_S| = |\mathfrak{M}|$, $\mathfrak{M} \restriction_\sigma = \mathfrak{M}_S \restriction_\sigma$, and for any $\exists$-formula $\varphi(x_0, x_1, \ldots, x_n) \in F_\sigma$

$$\mathfrak{M}_S \models \forall x_1 \ldots \forall x_n (\exists x \varphi(x, x_1, \ldots, x_n) \to$$

$$\to \varphi(f_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)).$$

**Theorem (S. 1996, with corr. 2013)**

*If* $\mathrm{Th}(\mathfrak{M})$ *is regular then* $\mathbb{HF}(\mathfrak{M})$ *has the uniformization property if and only if, for some well-defined existential Skolem expansion* $\mathfrak{M}_S$ *of* $\mathfrak{M}$,

$$\mathfrak{M}_S \equiv_{s\Sigma} \mathfrak{M}.$$

**Theorem (S. 2013)**

*If* $\mathfrak{M}$ *is quasiregular then* $\mathbb{HF}(\mathfrak{M})$ *has the uniformization property if and only if, for some well-defined existential Skolem expansion* $\mathfrak{M}_S$ *of* $\mathfrak{M}$,

$$\mathfrak{M}_S \equiv_{s\Sigma} \mathfrak{M}.$$

## Proprsition (S. 2013)

1. *If $\mathfrak{M}$ is quasiregular then $\mathbb{HF}(\mathfrak{M})$ has a universal $\Sigma$-function and the reduction property.*
2. *If $\mathfrak{M}$ is quasiregular and $\mathbb{HF}(\mathfrak{M})$ has the uniformization property, then $\mathbb{HF}(\mathfrak{M})$ is $\Sigma$-equivalent to the Moschovakis expansion $\mathfrak{M}^*$.*

### Proprsition (S. 1996)

*For $\mathbb{R}$ and $\mathbb{Q}_p$, there exist well-defined $s\Sigma$-definable Skolem expansions.*

Proof: use $\Sigma$-definable topology and topological properties of definable subsets.

### Corollary (S. 1996, indep. Korovina 1996 for $\mathbb{HF}(\mathbb{R})$ )

*$\mathbb{HF}(\mathbb{R})$ and $\mathbb{HF}(\mathbb{Q}_p)$ have the uniformization property and a universal $\Sigma$-function.*

For an arbitrary dense linear order $\mathbb{L} = \langle L, \leqslant \rangle$, define its *interval extension*

$$\mathcal{I}(\mathbb{L}) = \langle I, \leqslant, \subseteq \rangle$$

as follows. A nonempty set $i \subseteq L$ is called an *interval* in $\mathbb{L}$ if, for any $l_1, l_2, l_3 \in L$ such that $l_1, l_3 \in i$ and $l_1 \leqslant l_3$, from $l_1 \leqslant l_2 \leqslant l_3$ it follows that $l_2 \in i$.

Let $I$ be the set of all intervals in $\mathbb{L}$. Elements of $L$ can be considered as intervals of the form $[l, l]$, $l \in \mathbb{L}$.

The relation $\leqslant$ of structure $\mathbb{L}$ induces a partial order relation $\leqslant$ on set $I$. Namely, for elements $i_1, i_2 \in I$, we set $i_1 \leqslant i_2$ if and only if $l_1 \leqslant l_2$ for any $l_1 \in i_1$ and any $l_2 \in i_2$.

Let $\mathcal{B}(\mathbb{L})$ be the Boolean algebra generated by $\mathcal{I}(\mathbb{L})$.

$\mathbb{L} \models \mathrm{DLO}$ is called *continuous* if for any $A, B \subset L$ such that $A < B$ and $A \cup B = L$, either $A$ has the supremum or $B$ has the infimum.

**Theorem**

1. If $\mathbb{L}$ is continuous, then $\mathcal{I}(\mathbb{L})_{Morley} \equiv_{s\Sigma} \mathbb{L}$;
2. If $\mathbb{L}$ is continuous, then $\mathcal{B}(\mathbb{L}) \equiv_{s\Sigma} \mathbb{L}$.

The definition of an approximation space is given below in the most general form. However, in this paper we will consider only very special examples of such spaces, generated by interval extensions.

**Definition**

An *approximation space* is an ordered triple

$$\mathcal{X} = \langle X, F, \leqslant \rangle,$$

where $X$ is a topological $\mathrm{T}_0$-space, $F \subseteq X$ is a basic subset of *finite elements* and $\leqslant$ is a specialization order on $X$.

We denote by $a \prec x$ the fact that $a \in F$ and $a \leqslant x$.
Also, we will consider so called *structured* approximation spaces, i.e., we assume $F$ to be the domain of some structure $\mathcal{F}$.

### Definition

Let $\mathbb{L}$ be a dense linear order. *The space of temporal processes over* $\mathbb{L}$ is the approximation space

$$\mathcal{T}(\mathbb{L}) = (P(L) \setminus \{\varnothing\}, \mathcal{I}(\mathbb{L}), \subseteq),$$

where $P(L)$ is the set of all subsets of $L$ and $\subseteq$ is the standard set-theoretic inclusion relation on $P(L)$.

### Definition

Let $\mathbb{L}$ be a dense linear order. *The atomic space of temporal processes* over $\mathbb{L}$ is the approximation space

$$\mathcal{T}_0(\mathbb{L}) = (P(L) \setminus \{\varnothing\}, \mathbb{L}, \subseteq),$$

where $P(L)$ is the set of all subsets of $L$ and $\subseteq$ is the standard set-theoretic inclusion relation on $P(L)$.

Let $\sigma$ be a finite predicate signature containing, among other symbols, a binary predicate symbol $\leqslant$. We recall the definition of a formula of dynamic logic $DL_\sigma$. Namely, formulas of logic $DL_\sigma$ have variables of two types — for finite objects and for arbitrary, potentially infinite, objects that can only be accessed with the help of their finite fragments (approximations). We denote these sets by $FV$ and $SV$, respectively. For the formula $\theta$, the sets of its free variables of these two types are denoted by $FV(\theta)$ and $SV(\theta)$, respectively. If $\theta$ is a first-order logic formula of signature $\sigma$, then all its variables, including free ones, are considered to be finite. Variables denoted by uppercase letters $(S, P, \dots)$ are by default considered as variables of type $SV$.

## Definition

The set of $\Delta_0^{DL}$*-formulas of logic* $DL_\sigma$ is defined as the least set $R$ such that

1) if $\theta$ is a first-order logic formula of signature $\sigma$, then $\theta \in R$;
2) if $\theta \in R$, $S \in SV$, $a \in FV$, then $[a|S]\theta \in R$, $\langle a|S\rangle\theta \in R$;
3) if $\theta \in R$, $a, s \in FV$, then $[a|s]\theta \in R$, $\langle a|s\rangle\theta \in R$;
4) if $\theta_0, \theta_1 \in R$, then $\neg\theta_0 \in R$, $(\theta_0 \wedge \theta_1) \in R$, $(\theta_0 \vee \theta_1) \in R$ and $(\theta_0 \to \theta_1) \in R$.

## Definition

Let $\mathcal{X} = (X, F, \leqslant)$ be a structured approximation space over the structure $\mathcal{F} = (F, \sigma^{\mathcal{F}})$ of signature $\sigma$. The *satisfiability relation* on $\mathcal{X}$ for a formula $\varphi$ of logic $DL_\sigma$ and an evaluation $\gamma : SV(\varphi) \cup FV(\varphi) \to X$ with $\gamma(x) \in F$ for any $x \in FV(\varphi)$, denoted by $\mathcal{X} \models \varphi \restriction \gamma$, is defined by induction on the complexity of $\varphi$:

1) $\mathcal{X} \models [x|S]\theta(x) \restriction \gamma$ if, for all $a \prec \gamma(S)$, $\mathcal{X} \models \theta \restriction \gamma^x_a$;
2) $\mathcal{X} \models \langle x|S\rangle\theta(x) \restriction \gamma$ if there exists $a \prec \gamma(S)$ such that $\mathcal{X} \models \theta \restriction \gamma^x_a$;
3) $\mathcal{X} \models [x|s]\theta(x) \restriction \gamma$ if, for all $a \prec \gamma(s)$, $\mathcal{X} \models \theta \restriction \gamma^x_a$;
4) $\mathcal{X} \models \langle x|s\rangle\theta(x) \restriction \gamma$ if there exists $a \prec \gamma(s)$ such that $\mathcal{X} \models \theta \restriction \gamma^x_a$;
5) $\mathcal{X} \models (\exists S)\theta(S) \restriction \gamma$ if there exists $S_0 \in X$ such that $\mathcal{X} \models \theta \restriction \gamma^S_{S_0}$

and so on.

## Definition

An approximation space $\mathcal{X}_1$ is $\Delta^{DL}$-*reducible* to an approximation space $\mathcal{X}_2$ (denoted by $\mathcal{X}_1 \leqslant_{DL} \mathcal{X}_2$), if $\mathcal{X}_1$ as a structure is $\Delta_0^{DL}$-definable in the approximation space $\mathcal{X}_2$, and

1) the structure of finite elements $\mathcal{F}_1$ is $\Delta_0^{DL}$-definable in $\mathcal{X}_2$ inside $\mathcal{F}_2$,
2) there is an effective procedure that associates with every $\Delta_0^{DL}$-formula of space $\mathcal{X}_1$ a $\Delta_0^{DL}$-formula of space $\mathcal{X}_2$, which defines the corresponding predicate in this presentation of space $\mathcal{X}_1$ in space $\mathcal{X}_2$.

## Theorem

*If $\mathbb{L}$ is continuous, then approximation spaces $\mathcal{T}(\mathbb{L})$ and $\mathcal{T}_0(\mathbb{L})$ are effectively DL-equivalent:*

$$\mathcal{T}(\mathbb{L}) \equiv_{DL} \mathcal{T}_0(\mathbb{L}).$$

The basic relations of the temporal logic of J.F. Allen are formalized in dynamic logic as follows: for arbitrary temporal processes $P_1, P_2 \subseteq T$,

$P_1$ **before** $P_2$ corresponds to the relation $[i_1|P_1][i_2|P_2](i_1 \leqslant i_2)$;
$P_1$ **after** $P_2$ corresponds to the relation $[i_1|P_1][i_2|P_2](i_2 \leqslant i_1)$;
$P_1$ **while** $P_2$ corresponds to the relation $[i_1|P_1]\langle i_2|P_2\rangle(i_1 = i_2)$;
$P_1$ **overlaps** $P_2$ corresponds to the relation $\langle i_1|P_1\rangle\langle i_2|P_2\rangle(i_1 = i_2)$ (or, in the different interpretation, to the relation $\langle i_1|P_1\rangle\langle i_2|P_2\rangle((i_1 = i_2))\wedge \wedge($ "$i_1$ is a final subinterval of $P_1) \wedge$ ("$i_2$ is an initial subinterval of $P_2$))), etc.

R. Montague formalized the semantic meaning of verbs in English. We recall some examples of such formalization. First, here is his analysis of tense *Present Progressive*.

> The sentence (i.e., state) **John is walking** is true at time $p$ if and only if there is an open interval $i$ such that $p$ is a subinterval of $i$ and for all $t \in i$ state **John walks** is true in moment $t$.

Interval extensions for the first time were essentially used by American linguists M. Bennett and B. Partee. As an example, we consider a formal description of tense *Past Simple*.

The sentence (i.e., state) **John ate the fish** ($= \alpha$) is true on interval $i$, if $i$ is a point interval, $\alpha$ refers to the interval $i'$, and there exists an interval $i'' < i'$ such that $i'' < i$ and the state **John eats the fish** is true on $i''$.

For another example, consider the formal description of tense *Present Perfect*.

The sentence (i.e., state) **John has eaten the fish** ($= \alpha$) is true on interval $i$, if $i$ is a point interval, $\alpha$ refers to the interval $i'$, $i$ is a subinterval of $i'$ and there is an interval $i'' < i'$ such that either $i$ is the final point of $i''$, or $i'' < i$ and the state **John eats the fish** is true on $i''$.

It is easy to construct $\Delta_0^{DL}$-formulas of signature $\langle \leqslant, \subseteq \rangle$ describing the corresponding relations between these processes (or states) in the space of temporal processes $\mathcal{T}$. Namely,

$$p \subseteq \text{“\textbf{John is walking}”} \iff$$
$$\iff \langle i | \text{“\textbf{John walks}”} \rangle ((p \subseteq i) \wedge (\text{“}i \text{ is an open interval”})),$$

$$p \subseteq \text{“\textbf{John ate the fish}”} \iff [i | \text{“\textbf{John eats the fish}”}](i < p),$$

$$p \subseteq \text{“\textbf{John has eaten the fish}”} \iff [i | \text{“\textbf{John eats the fish}”}](i \leqslant p).$$

In the examples above we consider the states **John walks**, **John is walking**, **John eats the fish**, **John ate the fish** and **John has eaten the fish**, together with the point interval treated as the "present moment". Actually, in these examples it is shown how to define from *Present Simple* more complex tenses. Hence, by the results obtained above, the reasoning about the statements expressed by various combinations of tenses and aspects of English can be carried using some uniform and effective procedure.

The structure of tenses and aspects of verbs in Russian is rather different than that in English. Namely, with three tenses (*Present*, *Past* and *Future*), there are two aspects: *Perfect* and *Imperfect*. The main difficulty for the analysis of Russian verbs is that these two aspects are *independent* in sense there is no basic and no derivable one.

# Publications

- **Effective Model Theory and Generalized Computability**

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Thank You!