Reverse mathematics and Ramsey theorem for pairs

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Section 1

Reverse mathematics

Motivation

pougnioule's question :

pougnioule [Répondre par message privé]

Membre depuis : il y a dix mois
Messages: 25

Équivalence de théorèmes
il y a neuf mois

Bonjour
Après que des amis se soient demandés si deux théorèmes étaient équivalents, et après avoir un **peu** réflé**chi** à la question, je suis arrivé à la conclusion que la question n'avait pas de sens (pour la raison évidente que si l'on fixe des axiomes, les théorèmes que l'on en déduit ne sont basés que sur les axiomes, donc on pourrait, dans une démonstration utilisant un théorème, s'affranchir de ce théorème et le redémontrer.)

Pourtant des personnes s**û**rement très érudites (comparé à moi) ont écr**it** une page Wikipédia sur le théorème d'inversion locale [fr.wikipedia.org] et à la section "USAGES" au premier paragraphe tout à la fin écrivent :

" Le théorème d'inversion locale est utilisé soit sous sa forme d'origine, soit sous la forme du théorème des fonctions implicites, qui lui est équivalent au sens où chacun peut se déduire de l'autre."

Je voulais savoir si c'était une erreur (donc que la notion d'équivalence de théorème est dénuée de sens) ou pas. Et savoir ce que voulaient dire les auteurs de ce texte (Je me dis qu'il y a bien un sens).
Merci d'avance :)

Edité 1 fois. La dernière correction date de il y a neuf mois et a été effectuée par AD.

Répondre | Citer | Alerter un modérateur

Motivation

pougnioule's question :

After some friends had wondered if two theorems are equivalent, and after giving it some thoughts, I reached the conclusion that the question does not make sense (for the obvious reason that once axioms are fixed, the theorems we deduce are based on the axioms only. It follow that any proof using a theorem could be achieved without it, by re-demonstrating it when needed).

However, some erudite people (compare to me) wrote a Wikipedia page on the local inversion theorem. At the end of the usage section's first paragraph they wrote :

"The local inversion theorem is used either in its original form, or in the form of the implicit function theorem, to which it is equivalent, in the sense that they can be deduced from one another."

I would like to know if it is a mistake (that is, if the notion of theorem equivalence doesn't make sense) or not. And I would like to know what the authors of the above text meant (there must be a meaning).

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Motivation

Maxtimax's answer :

Maxtimax [Répondre par message privé]

Membre depuis : il y a quatre années
Messages: 4 000

Re: équivalence de théorèmes
il y a neuf mois

Tu as parfaitement raison; les théorèmes sont tous équivalents les uns aux autres logiquement, puisqu'ils se démontrent à partir des axiomes, et ta justification est tout à fait correcte ("je suppose l'un, bon maintenant je m'en fiche et je fais la preuve de l'autre").

Pour autant, la phrase de wikipedia n'est pas dénuée de sens, et il y a plusieurs sens que "ces deux théorèmes sont équivalents" peut prendre, je vais en décrire 2, et te dire duquel il est, je pense, ici question :

1- Le sens technique. Deux théorèmes prouvés à partir d'axiomes T sont forcément équivalents comme on l'a déjà vu. Maintenant, imaginons que j'enlève des axiomes de T, pour obtenir T'. Il se peut que tes théorèmes ne puissent plus se prouver dans T', mais que leur équivalence puisse, elle, toujours y être prouvée. Un exemple bien connu est l'axiome du choix et le lemme de Zorn par exemple; ou encore différents principes équivalents au postulat des parallèles d'Euclide pour un exemple plus géométrique. En ce sens, on peut parler de théorèmes équivalents, et si T, T' ne sont pas précisées, c'est qu'elles sont supposées être claires dans le contexte (par exemple si je dis "L'axiome du choix et le lemme de Zorn sont équivalents", on se doute que je n'énonce pas une trivialité de ZFC, mais un théorème un peu intéressant de ZF).
Une partie importante de la logique, et de la théorie des preuves consiste à étudier ces relations entre des théorèmes bien connus, mais dans des théories plus faibles (par exemple, le lemme de Zorn n'est plus si clairement équivalent à l'axiome du choix dans la théorie Z - d'ailleurs je ne sais plus s'il l'est ou pas)

2- Le sens pas technique. C'est un sens pédagogique ici : quand on étudie des maths, on a beaucoup de résultats de base qu'on utilise pour prouver des théorèmes plus gros. La plupart des résultats vus en cours se démontrent en 5minutes, tout au plus 10minutes - mais certains gros théorèmes prennent plus de temps, 30min, voire une heure, voire s'étalent sur plusieurs cours. Parfois il s'avère qu'on a plusieurs tels gros théorèmes, disons T_1 et T_2, mais que bien que la preuve de chacun des deux individuellement soit compliquée, il est facile de conclure T_2 à partir de nos résultats de base et de T_1 ; on dit en général que T_2 est un corollaire de T_1 ; quand on remarque que si on avait prouvé T_2 on pourrait aussi aisément en déduire T_1 (à partir de nos résultats de base) on a tendance à se dire que notre grosse preuve de quelques heures aurait aussi bien pu être faite pour T_2, en un sens on se dit que c'est arbitraire de prouver T_1 puis d'en déduire T_2 ou de faire l'inverse, que ça dépend des goûts de la personne qui prouve. Dans ces cas-là on aura tendance à dire que T_1 et T_2 sont équivalents : ce n'est pas faire un énoncé technique à propos de théories plus faibles, mais simplement un énoncé pédagogique qui nous dit que l'un n'est pas "plus compliqué" que l'autre.

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Maxtimax's answer :

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1. The technical sense. Two theorems proved from a theory T are necessarily equivalent as we said. Suppose now that I remove some axioms from T, in order to obtain a theory T'. Maybe the theorems cannot be proved within T' anymore, but maybe their equivalence can. A well known example is the axiom of choice together with Zorn's lemma. [...]

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2. The non-technical sense. This is the pedagogical sense : when we study math, we use a lot of basic results to prove bigger theorems. Most of the results seen in class can be shown in 5 minutes, at most 10 minutes - but some big theorems take longer, 30 minutes or one hour, sometimes even several sessions. Sometimes we have several big theorems, say T_1 and T_2, so that the proof of each of them is individually complicated, but such that it is easy to deduce T_2 from T_1 together with our basic results. We normally say that T_2 is a corollary from T_1. [...]

Motivation

Reverse mathematics provide an answer to `pougnioule`'s concerns, by giving a formal meaning to `Maxtimax`'s answer, that is, by giving a formal meaning to our intuition on sentences like

- Theorems A ans B are equivalent
- Theorem A does not follow from theorem B

Usual sense :

$$A \longleftrightarrow_{\mathrm{ZFC}} B$$

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Usual sense :

$$A \longleftrightarrow_{\not{\mathrm{ZFC}}} B$$

$$\ldots$$

New sense :

$$A \longleftrightarrow_{\mathrm{RCA}_0} B$$

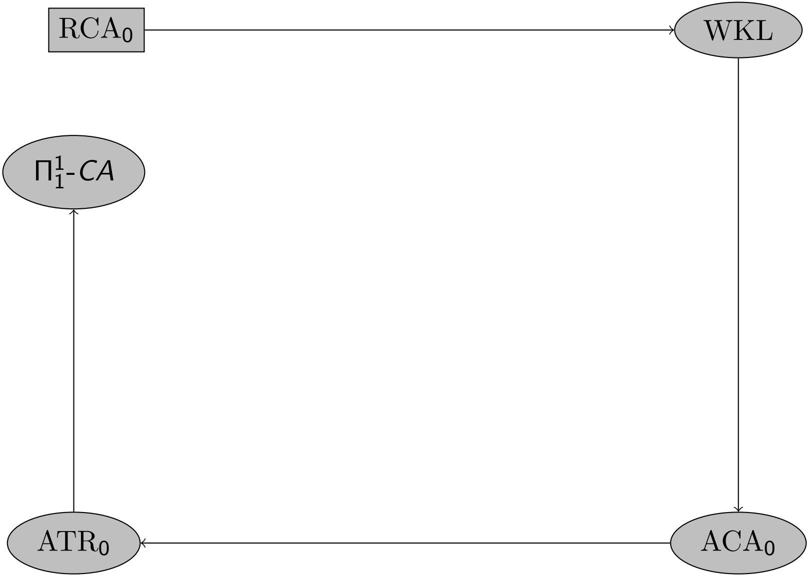
Concretely

Second order arithmetic

	First order elements :	**Second order elements :**
	Integers	Reals
Examples	$0, 1, 2, \ldots$	$\mathbb{N}, \pi, \sqrt{2}, \ldots$
Variables	$x, y, z, \ldots$	$X, Y, Z, \ldots$
Models	$\mathbb{N}$	Computable sets

During this talk, the models will always be ω-models : models in which integers are the true integers : only the second order part will change.

Five main axiomatic systems



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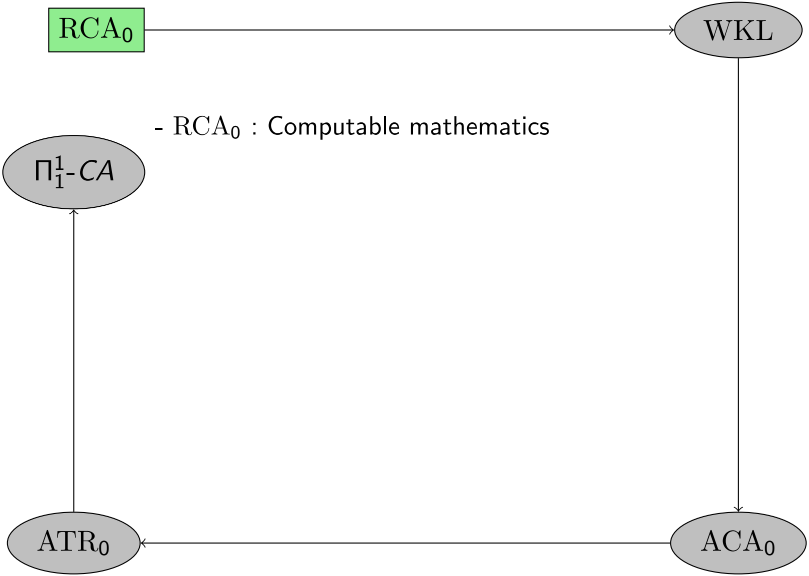
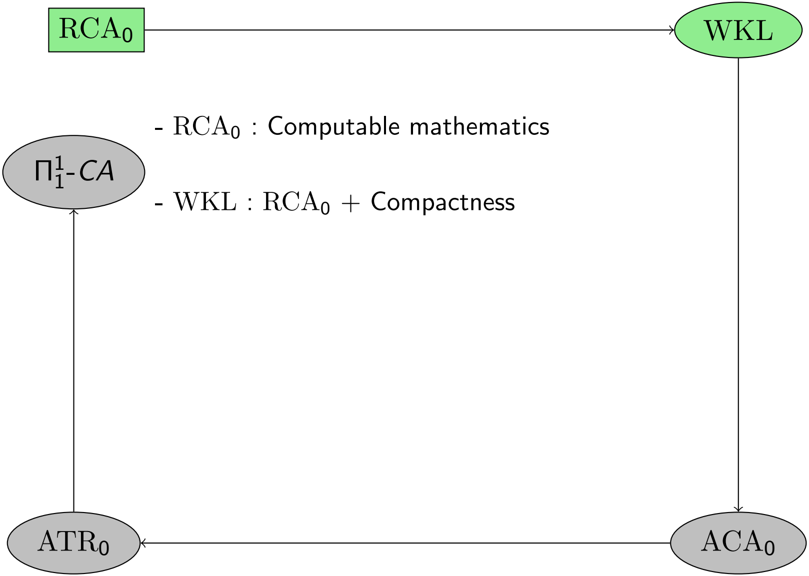


- RCA_0 : Computable mathematics

Five main axiomatic systems



- RCA_0 : Computable mathematics

- WKL : RCA_0 + Compactness

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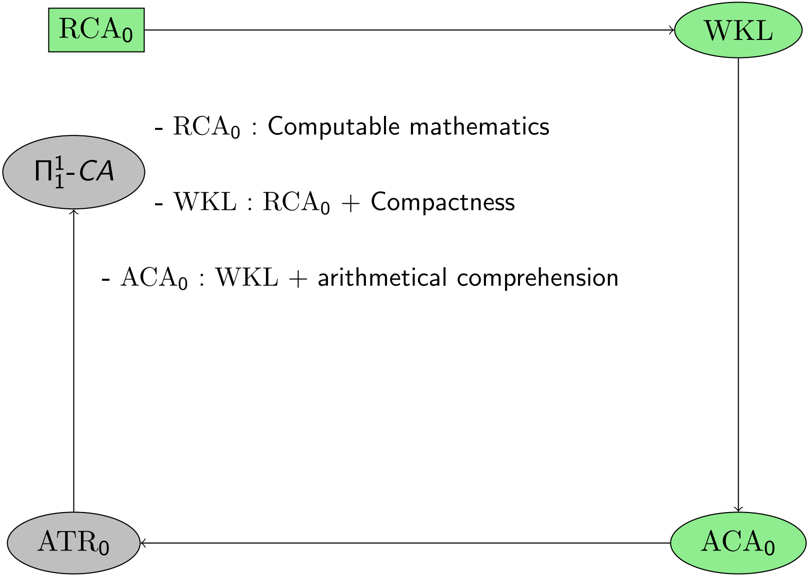

- RCA_0 : Computable mathematics
- WKL : RCA_0 + Compactness
- ACA_0 : WKL + arithmetical comprehension

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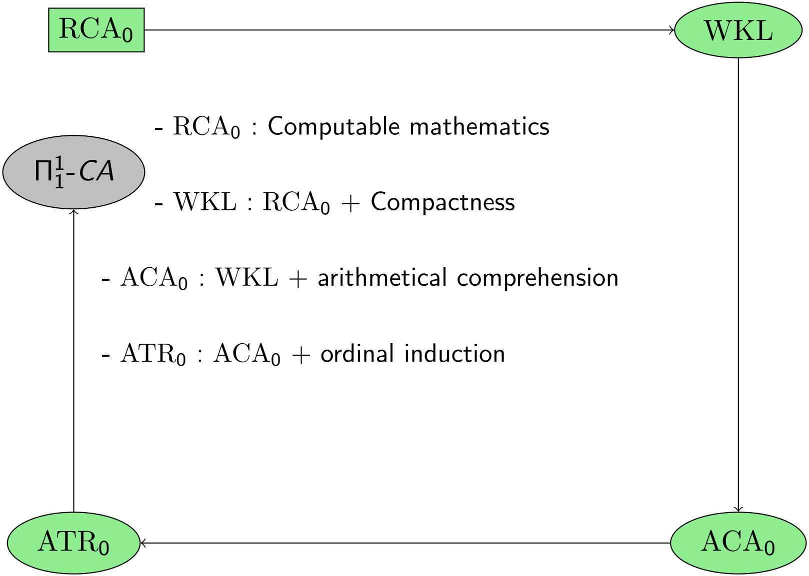

- RCA_0 : Computable mathematics
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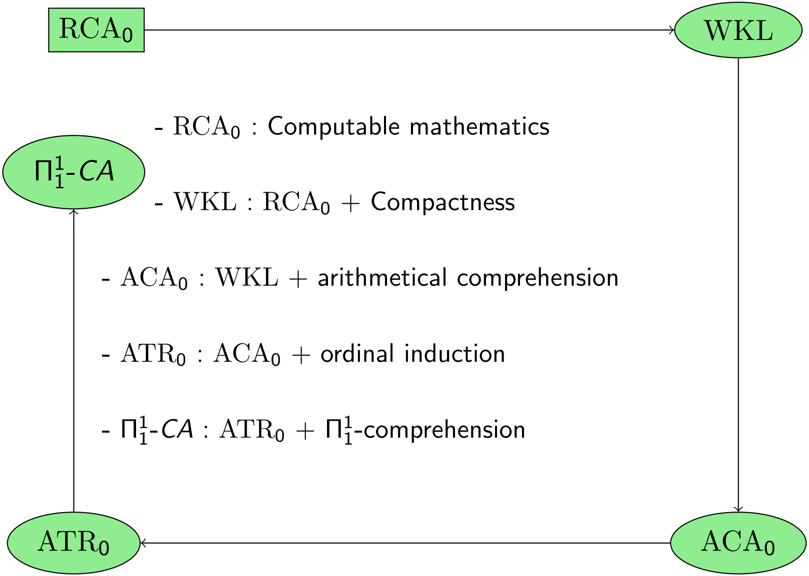
- ACA_0 : WKL + arithmetical comprehension
- ATR_0 : ACA_0 + ordinal induction

Five main axiomatic systems



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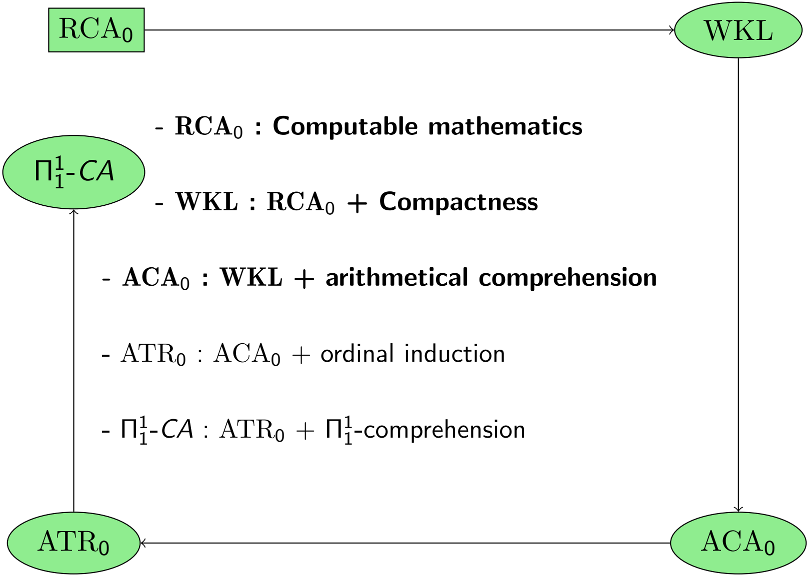
- ATR_0 : ACA_0 + ordinal induction
- Π^1_1-CA : ATR_0 + Π^1_1-comprehension

Five main axiomatic systems



- **RCA_0 : Computable mathematics**
- **WKL : RCA_0 + Compactness**
- **ACA_0 : WKL + arithmetical comprehension**
- ATR_0 : ACA_0 + ordinal induction
- Π^1_1-CA : ATR_0 + Π^1_1-comprehension

RCA_0 : Computable mathematics

RCA_0 axioms :

1. Robinson arithmetic
2. Induction on integers for Σ^0_1 formulas
3. Comprehension on sets of integers for Δ^0_1 formulas

A model of RCA_0 is closed by

- Turing reduction : If X belongs to the model, and X computes Y, then Y belongs to the model.
- Turing join : If X, Y belong to the model then $X \oplus Y$ is in the model.

RCA_0 theorems : examples

Theorem (RCA_0 - uncountability of the reals)

For every function $f : \mathbb{N} \to \mathbb{R}$, there exists $r \in \mathbb{R}$ such that $r \notin f(\mathbb{N})$.

Theorem (RCA_0 - Intermediate value theorem)

For every function $f : \mathbb{R} \to \mathbb{R}$ continuous on $[a, b]$, the set $f([a, b])$ is an interval.

Theorem (RCA_0 - Weak completeness theorem)

Every (countable) consistent theory which is closed by logical consequence has a model.

WKL_0 : Compactness

WKL_0 axioms

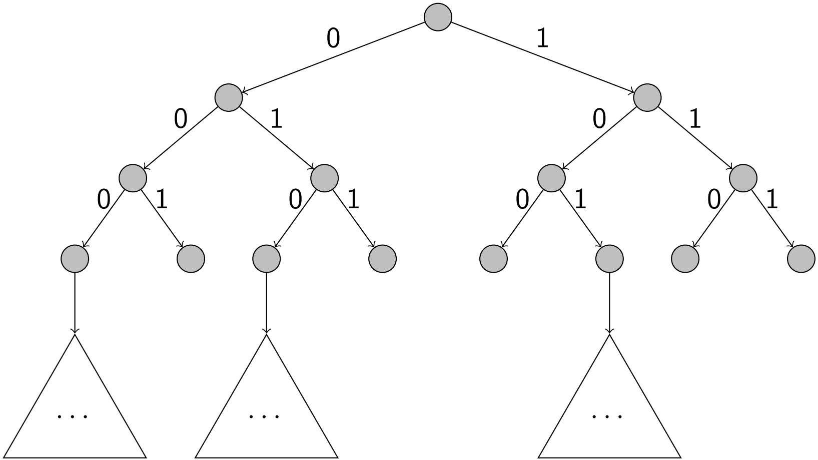
1. RCA_0 axioms
2. weak Köning's lemma : every infinite binary tree has an infinite path.

A model of WKL_0 is a **Scott set** :

- closed by Turing reduction : If X belongs to the model, and X computes Y, then Y belongs to the model.
- closed by Turing join : If X, Y belong to the model then $X \oplus Y$ is in the model.
- If an infinite binary tree T belongs to the model then an infinite path X of T belongs to the model.

WKL_0 : weak König's lemma

Which side has infinitely many nodes ?



WKL_0 : weak König's lemma

The following sentences are equivalent :

1. X computes an infinite path in every infinite binary tree.
2. X computes a complete and consistent extension of Peano arithmetic.
3. X computes a function $f : \mathbb{N} \to \{0, 1\}$ such that $\forall n\ f(n) \neq \Phi_n(n)$

Furthermore the functions

$$\{f : \mathbb{N} \to \{0, 1\}\ :\ \forall n\ f(n) \neq \Phi_n(n)\}$$

are the paths of some infinite computable binary tree.

→ There is a universal instance of weak König's lemma.

WKL_0 theorems

Theorem (WKL_0 - Heine/Borel Lemma)

From every covering of $[0,1]$ by open sets we can extract a finite subcovering.

Theorem (WKL_0 - Analysis)

Every continuous function in $[0,1]$ admits and reach a maximal value.

Theorem (WKL_0 - Algebra)

Every countable commutative ring contains a prime ideal.

Theorem (WKL_0 - Gödel's completeness theorem)

Every countable consistent theory has a model.

ACA_0 : comprehension

ACA_0 axioms :

1. WKL_0 axioms.
2. Comprehension for arithmetical formulas.

A model of ACA_0 is a set

- closed by Turing reduction : If X belongs to the model, and X computes Y, then Y belongs to the model.
- closed by Turing join : If X, Y belong to the model then $X \oplus Y$ is in the model.
- closed by the halting problem : If X belongs to the model, then X', the halting problem relative to X belongs to the model.

ACA_0 Theorems

Theorem (ACA_0 - Bolzano/Weierstrass)

Every infinite sequence of points in $[0,1]$ has a convergent subsequence.

Theorem (ACA_0 - Analysis)

Every increasing bounded sequence of reals has a limit.

Theorem (ACA_0 - Algebra)

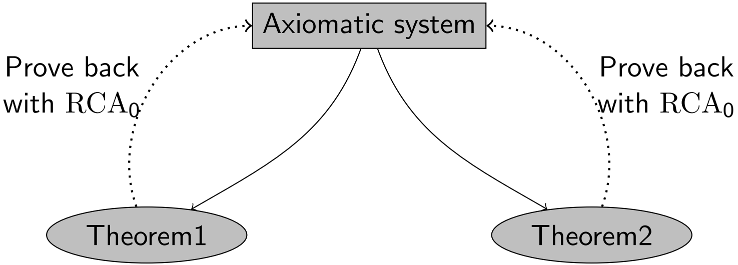
Every countable commutative ring contains a maximal ideal.

Theorem (ACA_0 - Ramsey's theorem)

Let $n > 2$. For every function $f : [\mathbb{N}]^n \to \{0,1\}$, there exists X with $|X| = \infty$ such that $|f([X]^\omega)| = 1$.

Why reverse mathematics ?

Find back the axioms from theorems



Goal

Find minimal axiomatic system to prove theorems.

Section 2

Ramsey theorem for pairs

Ramsey theorem for pairs

Definition

RT^n_m : For every coloring of the sets of integers of size n with m colors, there exists an infinite set whose every infinite subset of size n have the same color.

1. An **instance** I of RT^n_m is a function $c : [\mathbb{N}]^n \to \{0, \ldots, m\}$.
2. A **solution** of I is an infinite set X whose every subset of size n have the same color using c.

The **principle** RT^n_m says : Every **instance** of RT^n_m has a **solution**. The statement RT^n_m is provable in T if in every model M of T, for every instance $I \in M$ of RT^n_m, there exists a solution to I in M.

Ramsey theorem for pairs

Theorem (Jockush, 1972)

For every $n \geqslant 1$, Ramsey theorem for n-tuples - RT^n_2 - is provable in ACA_0.

For every n and every color $c : [\omega]^n \to \{0,1\}$, the set $c^{(n)}$ computes a solution of c.

Theorem (Specker, 1972)

Ramsey theorem for pairs - RT^2_2 - is not provable in RCA_0.

Construction of a computable function $c : [\omega]^2 \to \{0,1\}$ for which there exists no computable solution.

Ramsey theorem for pairs

Theorem (Jockush, 1972)

Ramsey theorem for pairs - RT^2_2 - is not provable in WKL_0.

Construction of a computable function $c : [\omega]^2 \to \{0, 1\}$ for which there is no Σ^0_2 solution.

Theorem (Jockush, 1972)

Ramsey's theorem for triplets - RT^3_2 - is equivalent to ACA_0.

Construction for every X of an X-computable function $c : [\omega]^3 \to \{0, 1\}$ every solution of which computes X'.

Ramsey theorem for pairs

Theorem (Seetapun, 1995)

Ramsey theorem for pairs - RT^2_2 *- does not prove* ACA_0*.*

For every $X \not\geq \emptyset'$, and every X-computable function $c : [\omega]^2 \to \{0, 1\}$, construction of a solution for c which does not compute $\emptyset'$.

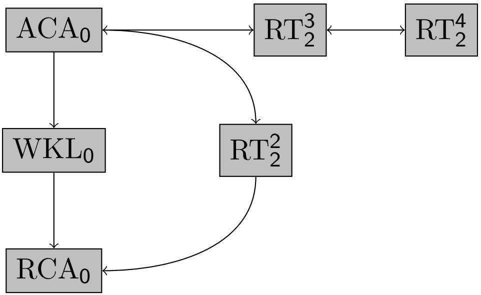
$\to$ construction of a model of $\mathrm{RT}^2_2 \oplus \mathrm{RCA}_0$ which is not a model of ACA_0.

Theorem (Liu, 2012)

RT^2_2 *does not prove* WKL_0*.*

Ramsey theorem for pairs

Summing up



Implications are strict

Ramsey theorem for pairs

Definition

A set C is $\{R_n\}_{n\in\omega}$-*cohesive* if $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$ for every n.

Definition

COH : For every sequence of sets $\{R_n\}_{n\in\mathbb{N}}$, there exists an $\{R_n\}_{n\in\mathbb{N}}$-cohesive set.

Definition

A coloring $c : [\mathbb{N}]^2 \to \{0,1\}$ is *stable* if $\forall x \ \lim_{y\in\omega} c(x,y)$ exists.

Definition

SRT^2_2 : Every stable color $c : [\mathbb{N}]^2 \to \{0,1\}$ admits an homogeneous set.

Ramsey theorem for pairs

Theorem (Cholak, Jockusch, Slaman and Mileti)

$$\mathrm{RT}^2_2 \leftrightarrow_{\mathrm{RCA}_0} \mathrm{COH} \oplus \mathrm{SRT}^2_2$$

$\mathrm{COH} \oplus \mathrm{SRT}^2_2 \rightarrow \mathrm{RT}^2_2$ (Cholak, Jockusch, Slaman)
Let $c : [\mathbb{N}]^2 \rightarrow \{0, 1\}$. Let $R_n = \{y \ : \ c(n, y) = 0\}$. Let C be an $\{R_n\}_{n\in\omega}$-cohesive set. Then c is stable on C.

$\mathrm{RT}^2_2 \rightarrow \mathrm{COH}$ (Mileti)
Construction of a computable coloring for which every solution is cohesive.

$\mathrm{RT}^2_2 \rightarrow \mathrm{SRT}^2_2$
Trivial

Ramsey theorem for pairs

Theorem (Liu)

For every non-PA set X, for every set A there exists $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that $G \oplus X$ is non-PA.

Liu's theorem is used to build a model of SRT^2_2 which is not a model of WKL, using the following equivalence

Definition

D^2_2 : Every Δ^0_2 instance of RT^1_2 has a solution.

$$\mathrm{D}^2_2 \leftrightarrow_{\mathrm{RCA}_0} \mathrm{SRT}^2_2$$

Given a stable color $c : [\omega]^2 \to \{0,1\}$, let A be the Δ^0_2 set such that $n \in A$ iff $\lim_x c(n,x) = 1$. From an infinite subset X of A or of $\overline{A}$, one can compute an infinite subset of X homogeneous for c. Using that COH does not imply WKL, we build a model of $\mathrm{SRT}^2_2 \oplus \mathrm{COH}$ (and then of RT^2_2) which is not a model of WKL.

Ramsey theorem for pairs

In the equivalence

$$\mathrm{RT}^2_2 \leftrightarrow_{\mathrm{RCA}_0} \mathrm{COH} \oplus \mathrm{SRT}^2_2$$

Do we need the two principles on the left ? In particular do we have

$$\mathrm{COH} \rightarrow_{\mathrm{RCA}_0} \mathrm{SRT}^2_2 \text{ ? or } \mathrm{SRT}^2_2 \rightarrow_{\mathrm{RCA}_0} \mathrm{COH} \text{ ?}$$

Answer :

Theorem (Hirschfeldt, Jockusch, Kjoss-hanssen, Lempp and Slaman)

$\mathrm{COH} \oplus \mathrm{RCA}_0 \nrightarrow \mathrm{SRT}^2_2$

Theorem (Chong, Slaman and Yang)

$\mathrm{SRT}^2_2 \oplus \mathrm{RCA}_0 \nrightarrow \mathrm{COH}$

Ramsey theorem for pairs

Theorem (Chong, Slaman and Yang)

$\mathrm{SRT}^2_2 \oplus \mathrm{RCA}_0 \nrightarrow \mathrm{COH}$

The proof of Chong, Slaman and Yang does not work in ω-models. It uses the fact that RCA_0 only has induction for Σ^0_1 formulas.

The separation within ω-models was only solved recently :

Theorem (M., Patey)

There exists an ω-model of $\mathrm{SRT}^2_2 \oplus \mathrm{RCA}_0$ which contains no cohesive set for primitive recursive functions, and therefore is not a model of COH.

Section 3

Splitting ω in two

The question

What can we encode inside every infinite subsets of both two halves of ω ?

A splitting :

Such that :

- **Each infinite subset** of the blue part has some comp. power
- **Each infinite subset** of the red part has some comp. power

Answer : Not much...

A precision

What if we drop the complement thing ?

Consider any set X. Then we can encode X into every infinite subset of a set A the following way : We let A be all the integers which correspond to an encoding of the prefixes of X (using some computable bijection between 2^ω and ω).

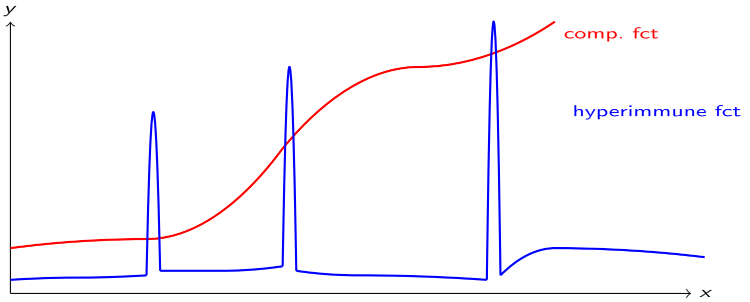
$$\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \ldots X$$

$$A(n) = 1 \text{ iff } n \text{ encodes } \sigma_n \text{ for some } n$$

Encoding Hyperimmunity

Definition (Hyperimmunity)

A set X is of *hyperimmune degree* if X computes a function $f : \omega \to \omega$, which is not dominated by any computable function.



Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that every $X \in [A^0]^\omega \cup [A^1]^\omega$ is of hyperimmune degree.

Encoding Hyperimmunity

Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that every $X \in [A^0]^\omega \cup [A^1]^\omega$ is of hyperimmune degree.

We split ω by alternating larger and larger blocks of consecutive integers in A^0 and A^1.

For X infinite subset of A^0 or A^1, the hyperimmune function is given by $f(n)$ to be the n-th number which appears in X.

Encoding DNC

Definition (Diagonally non-computable degree)

A set X is of *DNC degree* (diagonally non-computable) if X computes a function $f : \omega \to \omega$, such that $f(n) \neq \Phi_n(n)$ for every n.

Theorem

The following are equivalent for a set X :

- *X is of DNC degree.*
- *X computes a function which on input n can output a string of Kolmogorov complexity greater than n.*
- *X computes an infinite subset of a Martin-Löf random set.*

Encoding DNC

Definition (Informal definition of Kolmorogov complexity)

We say $K(\sigma) \geqslant n$ if the size of the smallest program which outputs σ is at least n.

Definition (Informal definition of Martin Löf randomness)

We say X is Martin Löf random if the Kolmogorov complexity of each of its prefix σ is greater than $|\sigma|$.

Theorem

X is of DNC degree iff X computes an infinite subset of a Martin-Löf random set.

$$
\begin{aligned}
& 0010111010100110110011010010110101100101010 10\ldots \\
\to\; & 0000100000000010000000000000001000110000000010\ldots \\
\to\; & 1111111110111111110111111011111111101111011110\ldots
\end{aligned}
$$

Encoding enumerating non-enumerable things

Theorem [Tennenbaum, Denisov]

There exists a computable order of ω, of order type $\omega + \omega^*$ which has no infinite ascending or descending c.e. sequence.

Consider $A \subseteq \omega$ the initial segment of order-type ω.

- Any infinite subset $X \subseteq A$ enumerates A (by enumerating things smaller than elements of X)
- Any infinite subset of $X \subseteq \overline{A}$ enumerates $\overline{A}$ (by enumerating things larger than elements of X)

Corollary [Tennenbaum, Denisov]

There exists a set A such that every set $G \in [A]^\omega \cup [\overline{A}]^\omega$ can make c.e. something which is not c.e.

Cone avoidance

Theorem [Dzhafarov and Jockusch]

Let $X \subseteq \omega$ be non-computable. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that $G \not\geq_T X$.

$\rightarrow$ (Seetapun) RT^2_2 does not prove ACA_0

The proof uses computable Mathias Forcing : Dzhafarov and Jockusch's technique has then been enhanced an reused in various manner by multiple authors to show other results of the same type, that we shall now expose.

More on cone avoidance

Theorem [Dzhafarov and Jockusch]

Let $X \subseteq \omega$ be non-c.e. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that X is not c.e. in G.

But we cannot avoid more than one c.e. set. On the other hand :

Theorem [Dzhafarov and Jockusch]

Let $\{X_n\}_{n \in \omega}$ be all non-computable. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G computes no X_n.

PA degrees

Definition

A set X is of P.A. degree if X computes a complete and consistent extension of Peano arithmetic.

Theorem

The following are equivalent :

- *X is of P.A. degree.*
- *X is diagonally non-computable with a $\{0,1\}$-valued function.*
- *X computes an infinite path in any non-empty Π^0_1 class.*

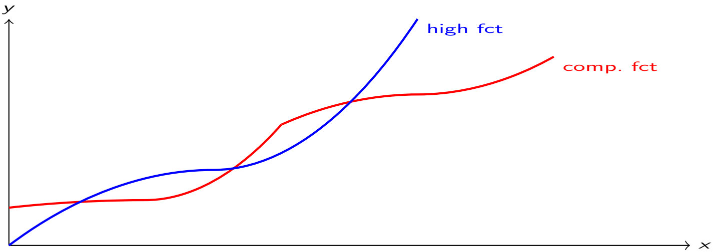
Theorem (Liu)

For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G is not of PA degree.

Non high

Definition

A set X is high if it computes a function which eventually grows faster than any computable function.



Theorem (M., Patey)

For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G is not high.

Non high

Theorem (Martin)

The following are equivalent for a set X :

- X *is high*
- $X' \geqslant_T \emptyset''$

Theorem (M., Patey)

Let X be non $\emptyset'$-computable. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that X is not G'-computable.

The proof uses of new forcing technique that builds upon Mathias forcing to control the second jump.

Iterating throught the ordinals

Theorem (M., Patey)

Let $\alpha < \omega_1^{ck}$. Let X be non $\emptyset^{(\alpha)}$-computable. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that X is not $G^{(\alpha)}$-computable.

Theorem (M., Patey)

Let X be non Δ^1_1. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that X is not $\Delta^1_1(G)$.

Theorem (M., Patey)

For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that $\omega_1^X = \omega_1^{ck}$.

Computing cohesive sets

Definition (Cohesiveness)

A set X if p-cohesive if for any primitive recursive set R_e we have $X \subseteq^* R_e$ or $X \subseteq^* \overline{R_e}$

Theorem (Folklore)

A set X computes a p-cohesive set iff X' is $\mathrm{PA}(\emptyset')$, that is, iff X' computes a function $f : \omega \to \{0,1\}$ such that $f(n) \neq \Phi_e^{\emptyset'}(e)$.

Theorem (M., Patey)

For every Δ^0_2 set A, there is an element $G \in [A]^\omega \cup [\overline{A}]^\omega$ such that G' is not $\mathrm{PA}(\emptyset')$.

Question

Is the former true for any set A?