Hilbert's Tenth Problem for Subrings of the Rational Numbers

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(Partially joint work with Ken Kramer.)

HTP: Hilbert's Tenth Problem

Definition

For a ring R, *Hilbert's Tenth Problem for R* is the set

$$HTP(R) = \{f \in R[X_0, X_1, \ldots] : (\exists \vec{a} \in R^{<\omega})\ f(a_0, \ldots, a_n) = 0\}$$

of all polynomials (in several variables) with solutions in R.

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Hilbert's original formulation in 1900 demanded a decision procedure for $HTP(\mathbb{Z})$.

Theorem (Matiyasevich-Davis-Putnam-Robinson, 1970)

$HTP(\mathbb{Z})$ is undecidable: indeed, $HTP(\mathbb{Z}) \equiv_1 \emptyset'$.

MDPR showed that $\emptyset'$ is *diophantine* in $\mathbb{Z}$, i.e., $\exists$-definable there.

Enumerating $HTP(\mathbb{Z})$: sums of three cubes

Theorem (Booker, March 2019)

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$(8{,}866{,}128{,}975{,}287{,}528)^3 + (-8{,}778{,}405{,}442{,}862{,}239)^3 + (-2{,}736{,}111{,}468{,}807{,}040)^3 = 33.$

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$(X_1^3 + X_2^3 + X_3^3 - 42)$ lies in $HTP(\mathbb{Z})$:
$(-80{,}538{,}738{,}812{,}075{,}974)^3 + (80{,}435{,}758{,}145{,}817{,}515)^3 + (12{,}602{,}123{,}297{,}335{,}631)^3 = 42.$

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Conjecture

$(X_1^3 + X_2^3 + X_3^3 - k) \in HTP(\mathbb{Z}) \iff k \not\equiv \pm 4 \bmod 9.$

This has been proven for all $k \leq 100$ in ω.
$\Longrightarrow$ holds for all k, since the only cubes in $\mathbb{Z}/(9)$ are 0 and ± 1.

Hilbert's Tenth Problem for $\mathbb{Q}$

Major Open Problem

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We always knew that $HTP(\mathbb{Q}) \leq_1 HTP(\mathbb{Z})$:

$$f(\vec{X}) \in HTP(\mathbb{Q}) \iff (\exists \vec{y}, z \in \mathbb{Z}) \left[f\left(\frac{y_1}{z}, \ldots, \frac{y_n}{z}\right) = 0 \ \&\ z > 0 \right]$$

$$\iff \left(Z^d \cdot f\left(\frac{Y_1}{Z}, \ldots, \frac{Y_n}{Z}\right)\right)^2 + \left(Z - \left(1 + \sum_{i=1}^{4} V_i^2\right)\right)^2 \in \mathrm{HTP}(\mathbb{Z}).$$

So the undecidability of $HTP(\mathbb{Z})$ focused attention on $HTP(\mathbb{Q})$.

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Proposition

For every subring $R \subseteq \mathbb{Q}$, we have $HTP(\mathbb{Q}) \leq_1 HTP(R)$, always via the same 1-reduction as shown above.

When is $HTP(R) \equiv_T HTP(\mathbb{Q})$?

Lemma (Shlapentokh, or Koenigsmann, following J. Robinson)

For each finite set $S_0 \subseteq \mathbb{P}$ of primes, the semilocal ring $R = \mathbb{Z}[\overline{S_0}^{-1}]$ satisfies $HTP(R) \equiv_1 HTP(\mathbb{Q})$, uniformly in S_0.

Here $R = \mathbb{Z}[W^{-1}]$, with $W = \mathbb{P} - S_0$ cofinite. But we can do better....

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Theorem (Eisenträger-M.-Park-Shlapentokh, 2017)

For every c.e. set C with $HTP(\mathbb{Q}) \leq_T C$, there is a computably presentable ring $R_W = \mathbb{Z}[W^{-1}]$ such that $HTP(R_W) \equiv_T C$ and the c.e. set $W \subseteq \mathbb{P}$ has lower density 0 in $\mathbb{P}$.

This means that $\liminf_{n\to\infty} \frac{|W \cap \{p_0,\ldots,p_n\}|}{n+1} = 0$. It is open whether this theorem can be strengthened to make the limsup equal 0 as well.

The construction by Eisenträger-M.-Park-Shlapentokh is purely computably theoretic: a finite-injury argument, mixed with coding of C.

HTP-generic subrings R_W of $\mathbb{Q}$

We satisfy, for each $f = f_0, f_1, f_2, \ldots \in \mathbb{Z}[X_1, X_2, \ldots]$,

$$\mathcal{P}_f: \quad f \notin HTP(R_W) \iff (\exists \text{ finite } S_f \subseteq \overline{W})\ f \notin HTP(R_{\mathbb{P}-S_f}).$$

At each stage s, some finite set $S_{f,s}$ of primes is *forbidden* to $\mathcal{P}_f$. Each $\mathcal{P}_f$ tries to find a rational solution $f(\vec{x}) = 0$ whose denominators are not divisible by the primes in $S_{f,s}$. If it finds one, it enumerates into W the prime factors of those denominators, so R_W contains this solution. The set $S_{f,s}$ is large enough to make the lower density approach 0, and changes only when $\mathcal{P}_f$ is injured by a higher-priority action.

The resulting R_W is *HTP-generic*: for every f, either:

- $f \in HTP(R_W)$, so we find a solution by enumerating R_W;
- or some finite $S_f = \lim_s S_{f,s}$ has $S_f \cap W = \emptyset$ and $f \notin HTP(R_{\mathbb{P}-S_f})$.

So, with an $HTP(\mathbb{Q})$-oracle, we can decide whether $f_0 \in HTP(R_W)$; then determine S_{f_1} from that and decide whether $f_1 \in HTP(R_W)$, etc.

Subrings of $\mathbb{Q}$

A subring R of $\mathbb{Q}$ is characterized by the set of primes p such that $\frac{1}{p} \in R$. For each set W of primes, set

$$R_W = \mathbb{Z}[W^{-1}] = \left\{ \frac{m}{n} \in \mathbb{Q} \ : \ \text{all prime factors of } n \text{ lie in } W \right\},$$

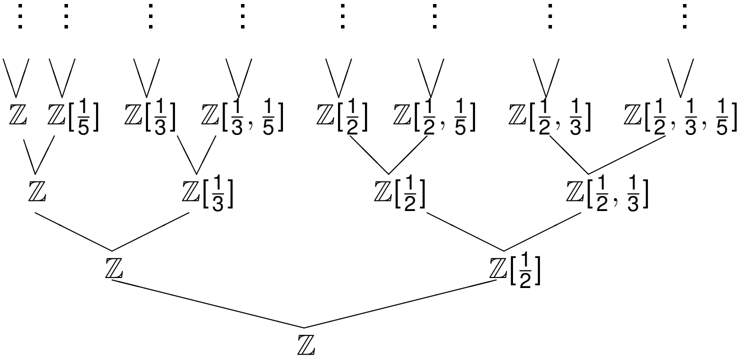
the subring generated by inverting all the primes in W.

We often move effectively between subsets V of ω and $W = \{p_n : n \in V\} \subseteq \mathbb{P}$, the set of primes which V describes.

Notice that $R_w = \mathbb{Z}[W^{-1}]$ is computably presentable iff W is c.e., while R_W is a computable subring of $\mathbb{Q}$ iff W is computable.

For $R \subseteq \mathbb{Q}$ we will treat $\{f \in \mathbb{Z}[\vec{X}] : (\exists \vec{x} \in R^{<\omega})\ f(\vec{x}) = 0\}$ as $HTP(R)$.

Subrings of $\mathbb{Q}$ as paths through a tree



In the signature of rings, the natural topology on the space of all subrings of $\mathbb{Q}$ is the Scott topology. In the extended signature with a unary predicate for invertibility, it is the Cantor topology. Either way, we have Lebesgue measure and also Baire category on the space.

HTP as an operator

HTP maps each subset $W \subseteq \mathbb{P}$ of the primes to $HTP(R_W)$, viewed as a subset of ω by coding. The most obvious analogy is between *HTP* and the jump operator $W \mapsto W'$. However, *HTP* is an *enumeration operator*: given any enumeration of W, it can enumerate $HTP(R_W)$, uniformly and effectively. The jump is not an enumeration operator.

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For an enumeration operator E, each set A can enumerate A', hence can enumerate $E(A')$. Therefore $E(A')$ is c.e. in A:

$$E(A') \leq_1 A',$$

so all sets of the form $W = A'$ have $E(W) \leq_T W <_T W'$. In particular, when $W = A'$, we have $HTP(R_W) \equiv_1 W$. So the HTP operator does not always increase complexity.

($W \leq_1 HTP(R_W)$ always holds: $p \in W \iff (pX - 1) \in HTP(R_W)$.)

How much difference between W' and $HTP(R_W)$?

Not many sets are jumps, but there is a widespread subtler difference:

Proposition

Let E be an enumeration operator. Then, for every relatively c.e. set W, we have $W' \not\leq_1 E(W)$.

Proof: *Relatively c.e.* means that there is a set $V <_T W$ such that W is V-c.e. Now $E(W)$ must also be V-c.e., so $E(W) \leq_1 V'$. However, with $W \not\leq_T V$, we have $W' \not\leq_1 V'$, and thus $W' \not\leq_1 E(W)$.

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Theorem (Jockusch for category; Kurtz for measure)

The relatively c.e. sets are co-meager and have measure 1 in 2^ω.

Corollary (M, 2017)

For almost all subrings $R = R_W \subseteq \mathbb{Q}$, the MDPR result fails: $W' \not\leq_1 HTP(R)$, and the set W' is c.e. in R but not diophantine in R.

1-reductions vs. Turing reductions

We really want to compare W' and $HTP(R_W)$ under Turing reducibility. Then we could apply:

Theorem (M, 2016, 2020)

For any set $C \subseteq \omega$ (such as $\emptyset'$), the following are equivalent:

1. $HTP(\mathbb{Q}) \geq_T C$.
2. $HTP(R) \geq_T C$ for all subrings R of $\mathbb{Q}$.
3. $HTP(R) \geq_T C$ for a non-meager set of subrings R.

Additionally, $HTP(\mathbb{Q})$ is low $\iff$ a non-meager set of subrings R_W all satisfy $(HTP(R_W))' \leq_T W'$.

It is open whether a similar equivalence holds for Lebesgue measure.

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It is open whether a similar equivalence holds for Lebesgue measure.

Almost all sets W are *generalized low*$_1$, meaning that $W' \equiv_T \emptyset' \oplus W$. So the following equivalence holds on a comeager set of measure 1:

$$\emptyset' \leq_T HTP(R_W) \iff W' \leq_T HTP(R_W).$$

Trying to compute the jump

Generalized lowness shows that it is possible for an enumeration operator to compute the jump in most cases:
Let $E(W) = \emptyset' \oplus W$. Then $W' \leq_T E(W)$ on a comeager set, and on a set of measure 1, although of course it is far less common to have $W' \leq_1 E(W)$.

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Possibly this could hold of the *HTP* operator, but it cannot be uniform:

Theorem

For each enumeration operator E and each Turing functional Ψ, the set

$$\{W \subseteq \omega : W' \neq \Psi^{E(W)}\}$$

has positive measure. Thus it is impossible for $E(W)$ to compute W' uniformly on a set of measure 1.

Proving the theorem

Fix e such that $e \in A'$ iff A omits a long interval:

$$\Phi_e^A(n) = \begin{cases} 0, & \text{if } (\exists m > 0)\ \{m, m+1, \ldots, 2m\} \cap A = \emptyset; \\ \uparrow, & \text{otherwise.} \end{cases}$$

At least half of all A have $e \notin A'$, so there is some σ such that

$$\mu(\{A \subseteq \omega : \sigma \sqsubseteq E(A)\ \&\ \Psi^\sigma(e)\downarrow = 0\}) > 0.$$

Fix a finite S_0 with $\sigma \sqsubseteq E(S_0)$ for which $\mathcal{W} = \{A : S_0 \subseteq A\ \&\ \sigma \sqsubseteq E(A)\}$ has measure > 0. Then fix an $m > \max(S_0 \cup \{0\})$. Set

$$\mathcal{V} = \{B \in 2^\omega : (\exists A \in \mathcal{W})\ B = A - \{m, m+1, \ldots, 2m\}\}.$$

Now $\mu(\mathcal{V}) \geq \frac{\mu(\mathcal{W})}{2^{m+1}} > 0$, and $\Psi^{E(B)}(e) = \Psi^\sigma(e)\downarrow = 0 \neq B'(e)$ for $B \in \mathcal{V}$.

HTP-complete sets

Definition

A set W is *HTP-complete* if $W' \leq_1 HTP(R_W)$. (In particular, this holds if W' is diophantine in R_W.)

We have seen that HTP-completeness is uncommon, in terms of Lebesgue measure and Baire category. However, it does occur widely.

Theorem

For every set $C \subseteq \omega$, there is an HTP-complete set $W \equiv_T C$.
It follows that every Turing degree $\boldsymbol{d} \geq \mathbf{0}'$ contains a set of the form $HTP(R_W)$.

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Corollary

There exists a computable subring $R \subseteq \mathbb{Q}$ with $HTP(R) \equiv_1 \emptyset'$.

Of course, MDPR proved this in 1970, for $R = \mathbb{Z}$ specifically. But our proof will be much simpler than theirs!

One useful polynomial

Define $f(X, Y, \ldots) = (X^2 + Y^2 - 1)^2 + (\text{“ } X > 0 \text{ ”})^2 + (\text{“ } Y > 0 \text{ ”})^2$.

Solutions to $f = 0$ correspond to nonzero pairs $(\frac{a}{c}, \frac{b}{c})$ with $a^2 + b^2 = c^2$.

If $2 \mid c$, then $a^2 + b^2 \equiv 0 \bmod 4$, so $a^2 \equiv b^2 \equiv 0 \bmod 4$, so a, b, and c had a common factor of 2.
If an odd prime p divides c, then $a^2 \equiv -b^2 \bmod p$, and so -1 is a square modulo p. Hence $p \equiv 1 \bmod 4$.

But if $p \equiv 1 \bmod 4$, then $p = m^2 + n^2$ for some $m, n \in \mathbb{Z}$, and then

$$\left(\frac{m^2 - n^2}{p}\right)^2 + \left(\frac{2mn}{p}\right)^2 = \frac{(m^4 - 2m^2n^2 + n^4) + 4m^2n^2}{p^2} = \frac{(m^2 + n^2)^2}{p^2} = 1.$$

So $f \in HTP(R_W) \iff W$ contains some $p \equiv 1 \bmod 4$.

Many useful polynomials (joint with Ken Kramer)

The $f(X, Y)$ above is useful, but it is only one polynomial, and can code only one bit of information in $HTP(R_W)$. We need more:

Lemma (Kramer)

For an odd prime q, let $f_q(X, Y) = X^2 + qY^2 - 1$ (modified to make $Y > 0$). Then in every solution $(\frac{a}{c}, \frac{b}{c}) \in \mathbb{Q}^2$ to $f_q = 0$, all prime factors p of c satisfy $(\frac{-q}{p}) = 1$, i.e., $-q$ is a square mod p.

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Conversely, for any such p, $\mathbb{Z}[\frac{1}{p}]$ contains a nontrivial solution to $f_q = 0$.

Definition

The *q-appropriate primes p* are those for which $(\frac{-q}{p}) = 1$.

So q-appropriateness is decidable uniformly in q. Asymptotically, just half of the primes are q-appropriate. E.g., when $q \equiv 3 \bmod 4$,

$$\left(\frac{-q}{p}\right) = \left(\frac{q}{p}\right) \cdot \left(\frac{-1}{p}\right) = \left(\frac{p}{q}\right).$$

Coding C' into $HTP(R_W)$

For an arbitrary C-oracle, we build $W \leq_T C$ with $C' \leq_1 HTP(R_W)$. Since $W' \leq_1 C'$, this W will be HTP-complete.

Write $C' = \{e_0, e_1, e_2, \ldots\} \subseteq \mathbb{N}$. We build $W \subseteq \mathbb{P}$ in stages. At stage s, to code that $e_s \in C'$, we wish to make the polynomial $f_{q_{e_s}}$ lie in $HTP(R_W)$, which requires putting a q_{e_s}-appropriate prime p into W_{s+1}. Choose $p > \max(W_s)$ such that, for every $j \leq s$ with $j \neq e_s$, p is NOT q_j-appropriate.

Enumerating W in order makes $W \leq_T C$. (Also $C' \leq_1 HTP(R_W)$ will make $C \leq_T W$.) The second condition tries to ensure, for those $j \notin C'$, that no q_j-appropriate prime ever enters W. From stage j onwards, it succeeds. But what if some q_j-appropriate prime had already entered W before that?

Why does this work?

Here are the necessary lemmas for the construction to succeed.

Lemma (Shlapentokh, or Koenigsmann, following J. Robinson)

For each finite set $S_0 \subseteq \mathbb{P}$, the semilocal subring $\mathbb{Z}[\overline{S_0}^{-1}]$ is diophantine in $\mathbb{Q}$, and its diophantine definition there is uniform in S_0.

This allows us to ask $HTP(R_W)$ whether R_W contains a solution to f_{q_j} that does NOT require inverting any of the primes that had already entered W by stage j.

Lemma

For every finite set $S_0 \subseteq \mathbb{P}$ and every prime $q \notin S_0$, there exist infinitely many primes that are q-appropriate but (for all $q' \in S_0$) not q'-appropriate.

Thus we can always find a prime satisfying the two conditions. Recall: p is q-appropriate iff $-q$ is a square modulo p.

HTP and Turing reducibility

This result also shows how *HTP* can fail to preserve Turing reducibility, and can even reverse it. For a c.e. set W of non-low degree, we will have $HTP(R_W) \leq_T \emptyset'$, because *HTP* is an enumeration operator. But there will be a non-low set $C <_T W$, and a $V \equiv_T C$ with $HTP(R_V) \equiv_1 C' >_T \emptyset'$.

Thus $V <_T W$, yet $HTP(R_W) <_T HTP(R_V)$.

(This was the original joint work with Ken Kramer.)

Analogies between operators

In the foregoing construction we used the *boundary rings* of the polynomials f_q.

Definition

For a pseudojump operator E and an $x \in \omega$, a set W lies in the *boundary for x* if we have $x \notin E(W)$ but, for every $\sigma \sqsubseteq W$, some $\tau \sqsupseteq \sigma$ has $x \in E(\tau^{-1}(1))$. That is, no finite portion of W rules out the possibility that x might yet lie in $E(W)$.

The *E-generic sets* are those that $(\forall x)$ do not lie in the boundary for x.

For *HTP*, a ring in which no q-appropriate primes are inverted is a boundary ring for f_q. However, the set of all such rings has measure 0.

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Open Question

What is the measure of the set of HTP-generic rings? Does there exist a polynomial f whose boundary rings form a set of positive measure?

Boundary sets

For the jump operator, one readily builds a functional Φ_e for which $\{$boundary sets for e under the jump$\}$ has measure $> 1 - \epsilon$:

$$\Phi_e^A(n) = \begin{cases} 0, & \text{if } (\exists m > -\log_2(\epsilon))\ \{m+1, m+2, \ldots, 2m\} \subseteq A; \\ \uparrow, & \text{otherwise.} \end{cases}$$

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- If *HTP* has large boundary sets, there is more opportunity to code undecidable information, making $HTP(R_W) >_T W \oplus HTP(\mathbb{Q})$.
- If all *HTP* boundary sets have measure 0, then measure-1-many W are *HTP*-generic, with $HTP(R_W) \equiv_T HTP(\mathbb{Q}) \oplus W$, and

$$HTP(\mathbb{Q}) \geq_T C \iff \mu(\{W : HTP(R_W) \geq_T C\}) > 0.$$

- If all *HTP* boundary sets have measure 0, then there is no existential definition of $\mathbb{Z}$ in the field $\mathbb{Q}$. (Miller, 2017.)