Rationality of growths of groups

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Dehn's three problems

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- Kharlampovich '81: solvable group of derived length 3

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- Focus on finitely-generated groups with a computable representative system

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Goal: find “nice" *geodesic representative systems*

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A sequence satisfies a finite-depth recursion if and only if its associated series is rational.

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A finitely-generated group has growth function, i.e. $s(r) = |S_r|$, *being bounded above by a polynomial if and only if it is virtually-nilpotent.*

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Theorem (Valiunas, '19)

If G has rational growth, then $Cr^{\alpha}\lambda^{r} < |S_r| < Dr^{\alpha}\lambda^{r}$ for some $\alpha \in \mathbb{N}$, $\lambda \in [1, \infty)$, and $0 < C < D$.

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Generating sets that admit rational growths are "better" than others.

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abelian groups → metabelian groups → solvable groups

metabelian groups

metabelian: solvable with derived length 2

Theorem (Kharlampovich, '81)

There is a solvable group with derived length 3 whose word problem is not computable.

HNN extensions

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Theorem (Ho, Sánchez, in progress)

Any higher Baumslag-Solitar group has rational growth in the standard generating set.

Semidirect products

Parry '07: $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with $A = \begin{bmatrix} 0 & -1 \\ 1 & 2k \end{bmatrix}$ has rational growth.

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Conjecture

Every $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with $A \in SL(2, \mathbb{Z})$ has rational growth in some generating set.

Example: solvable Baumslag-Solitar groups

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