A Σ^1_1 axiom of finite choice and Steel forcing

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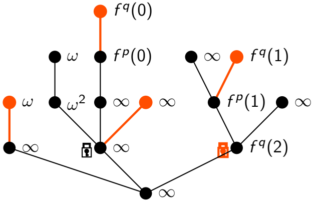


Q: *Given a theory, does it have a minimum model?*

Theorem (Grzegorczyk, Mostowski, Ryll-Nardzewski and Gandy, Kreisel, Tait)

The intersection of all ω-models of full second-order arithmetic is the class HYP of hyperarithmetical sets.

Q: *Is there a theory whose minimum ω-model is HYP?*

There are several such theories, including Δ^1_1-CA and Σ^1_1-AC.

In fact, every ω-model of (say) Σ^1_1-AC is hyp closed, i.e., closed under hyperarithmetic reduction and $\oplus$.

Q: *Is there a theory whose ω-models are exactly those which are hyp closed?*

ω-models of theories of second-order arithmetic

Q: *Is there a theory whose ω-models are exactly those which are hyp closed?*

Theory	Closure of ω-models	Minimum ω-model
RCA_0	Turing reduction and $\oplus$	REC
ACA_0	Arithmetic reduction and $\oplus$	ARITH
?	Hyp reduction and $\oplus$	HYP

The answer is no:

Theorem (van Wesep '77)

For any theory T all of whose ω-models are hyp closed, there is some T' which is strictly weaker than T, all of whose ω-models are hyp closed.

Theories of hyp analysis

Definition (Montalbán '06, relativizing Steel '78)

T is a theory of hyp analysis if:

1. every ω-model of T is hyp closed;
2. for every $Y \subseteq \omega$, $\mathrm{HYP}(Y) \models T$.

By van Wesep, there is no weakest theory of hyp analysis.

> **Q:** *How does the "zoo" of theories of hyp analysis look like?*

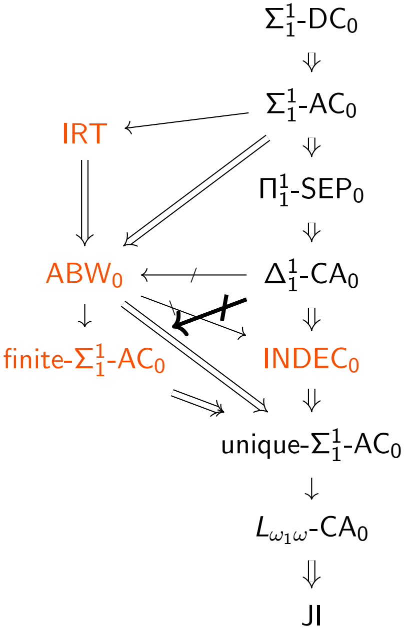
For example, are they linearly ordered?

> **Q:** *Are there any theories of hyp analysis which can be formulated without using concepts from logic?*

"Clearly" Σ^1_1-AC and Δ^1_1-CA do not qualify.

Over $\mathsf{RCA}_0 + I\Sigma^1_1$:

Kleene '59, Kreisel '62, Friedman '67, Harrison '68, van Wesep '77, Steel '78, Simpson '99, Montalbán '06, '08, Neeman '08 Conidis '12 Barnes, G., Shore in preparation



All of these separations (except $\Sigma^1_1\text{-}\mathsf{AC}_0 \nvdash \Sigma^1_1\text{-}\mathsf{DC}_0$) were proved using Steel forcing and variants thereof.

Kreisel '62: Σ^1_1-AC_0 consists of the sentences

$$(\forall n)(\exists X)\varphi(n, X) \to (\exists \langle X_n \rangle_n)(\forall n)\varphi(n, X_n)$$

for φ arithmetical.

Definition

Finite-Σ^1_1-AC_0 consists of the sentences

$$(\forall n)(\exists \text{ finitely many } X)\varphi(n, X) \to (\exists \langle X_n \rangle_n)(\forall n)\varphi(n, X_n)$$

for φ arithmetical.

Finite-Σ^1_1-AC_0 is a theory of hyp analysis, since it is sandwiched between theories of hyp analysis.

Our results

Theorem (G.)

There is an ω-model which satisfies Δ^1_1-CA_0 but not finite-Σ^1_1-AC_0.

Theorem (G.)

$\mathsf{ABW}_0 + I\Sigma^1_1 \vdash$ finite-Σ^1_1-AC_0.

Our results strengthen

Theorem (Conidis '12)

There is an ω-model which satisfies Δ^1_1-CA_0 but not ABW_0.

Theorem (Conidis '12)

$\mathsf{ABW}_0 + I\Sigma^1_1 \vdash$ unique-Σ^1_1-AC_0.

We do not know if finite-Σ^1_1-AC_0 implies ABW_0.

Steel's proof

Theorem (Steel '78)

There is an ω-model which satisfies Δ^1_1-CA_0 but not Σ^1_1-AC_0.

Steel constructs a generic tree $T^G \subseteq \omega^{<\omega}$ and generic paths $\langle \alpha_i^G \rangle_{i \in \omega}$ on T^G such that the α_i^G's are not easily definable from one another.

For each finite $F \subset \omega$, the model M_F consists of all sets which are computable in the λ^{th} jump of $T^G \oplus \langle \alpha_i^G \rangle_{i \in F}$, for some $\lambda < \omega_1^{CK}$.

Lemma

For each finite $F \subset \omega$, the set of paths on T^G in M_F is exactly $\{\alpha_i^G : i \in F\}$.

Finally, define $M_\infty = \bigcup_{F \subset \omega \text{ finite}} M_F$.

$\left(\bigcup_{F\subset\omega \text{ finite}} M_F =\right) M_\infty \models \neg\Sigma^1_1\text{-AC}_0$.

Consider $\varphi(n, X)$: X is a set of n distinct paths on T^G.
For each n, $\varphi(n, \cdot)$ has a solution in M_∞.

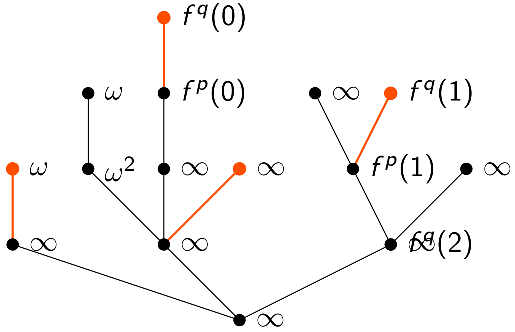
A Σ^1_1-AC$_0$-solution $\langle X_n\rangle_{n\in\omega}$ would compute an infinite sequence of distinct paths on T^G. But M_∞ does not contain any infinite sequence of distinct paths on T^G, by the lemma.

$M_\infty \models \Delta^1_1\text{-CA}_0$.

Main ingredient of proof is to show that if two forcing conditions are sufficiently "alike", then they force the same Σ^1_1 formulas.

This helps to control the complexity of the forcing relation.

Steel tagged tree forcing



A condition p consists of:

1. a finite tree T^p
2. finitely many paths $f^p(i)$
3. tags $h^p : T^p \to \omega_1^{CK} \cup \{\infty\}$

Steel tagged tree forcing

Conditions are $p = \langle T^p, f^p, h^p \rangle$ where:

1. $T^p \subseteq \omega^{<\omega}$ is finite
2. f^p is a finite partial function from ω to T^p (for each i, $f^p(i)$ is an initial segment of the generic path α_i^G)
3. h^p tags nodes of T^p with a computable ordinal or ∞
   - if $\tau \subseteq \sigma$, then $h^p(\tau) > h^p(\sigma)$
   - nodes in $\mathrm{range}(f^p)$ must be tagged ∞

q extends p if:

1. $T^q \supseteq T^p$
2. $h^q \supseteq h^p$
3. f^q can extend paths in f^p, subject to a technical restriction
4. f^q can add new paths to f^p, subject to a technical restriction ($f^q(j)$ is new if $j \notin \mathrm{dom}(f^p)$.)

Steel tagged tree forcing with locks (G.)

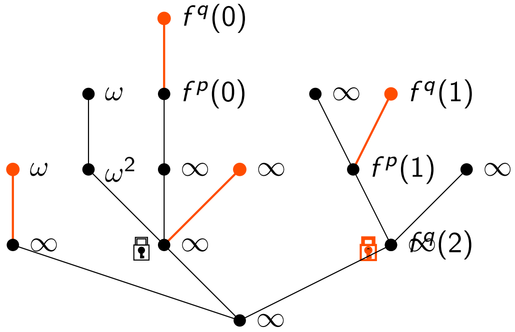
Conditions are $p = \langle T^p, f^p, h^p, \ell^p \rangle$ where:

1. $T^p \subseteq \omega^{<\omega}$ is finite
2. f^p is a finite partial function from ω to T^p
   (for each i, $f^p(i)$ is an initial segment of the generic path α_i^G)
3. h^p tags nodes of T^p with a computable ordinal or ∞
   - if $\tau \subseteq \sigma$, then $h^p(\tau) > h^p(\sigma)$
   - nodes in $\mathrm{range}(f^p)$ must be tagged ∞
4. $\ell^p \subseteq \{n : \langle n \rangle \in T^p\}$. $\langle n \rangle$ is *locked* if $n \in \ell^p$.

q extends p if:

1. $T^q \supseteq T^p$
2. $h^q \supseteq h^p$
3. f^q can extend paths in f^p, subject to a technical restriction
4. f^q can add new paths to f^p, subject to a technical restriction
   and restriction by the locks
5. $\ell^q \supseteq \ell^p$.

Steel tagged tree forcing with locks (G.)



A condition p consists of:

1. a finite tree T^p
2. finitely many paths $f^p(i)$
3. tags $h^p : T^p \to \omega_1^{CK} \cup \{\infty\}$
4. locks on certain nodes at level 1 of T^p

The locks ensure that

"X is a path on T^G which passes through $\langle n \rangle$" is an instance of finite-Σ^1_1-AC_0 in M_∞, i.e., for each n,

- M_∞ contains a path through $\langle n \rangle$;
- M_∞ contains only finitely many paths which pass through $\langle n \rangle$.

Locks impose the following restrictions on new paths:

Say $q \leq p$. If q adds a new path passing through $\langle n \rangle$, then either:

- none of the paths in p pass through $\langle n \rangle$, or
- $\langle n \rangle$ is unlocked in p.

The first condition ensures that some α_i^G passes through $\langle n \rangle$.

The second condition ensures that only finitely many paths α_i^G pass through $\langle n \rangle$.

Towards $M_\infty \models \Delta^1_1$-$\mathsf{CA}_0$: retagging

The tags in the forcing conditions help us control the paths on T^G. Recall from an earlier slide:

Lemma

For each finite $F \subset \omega$, the set of paths on T^G in M_F is exactly $\{\alpha_i^G : i \in F\}$.

Proof idea: Towards a contradiction, suppose p forces that some other S in M_F is a path on T^G.

We modify p to kill off S, i.e., retag some node in S from ∞ to an ordinal. Care is needed to not kill off any α_i^G, $i \in F$.

To obtain a contradiction, we want to show that the modified condition still forces that S is a path on T^G.

The formula asserting that S is a path on T^G is simple enough that we can prove that.

Comments on our forcing language

The model M_∞ is both "tall" and "wide":

- each M_F contains the λ^{th} jump of $T^G \oplus \langle \alpha_i^G \rangle_{i \in F}$ for every $\lambda < \omega_1^{CK}$;
- M_∞ contains α_i^G for every $i \in \omega$.

Correspondingly, formulas in our forcing language can be complicated in two ways:

- they might quantify over unranked set variables, which are allowed to range over all sets computable in $(\alpha_i^G)^{(\lambda)}$ for every $\lambda < \omega_1^{CK}$;
- they might quantify over all ranked sets in M_∞, instead of all ranked sets in some M_F;
- they might do both!

Retagging and the forcing relation

Steel's notion of retagging is sufficient for controlling the forcing relation for "simple" formulas:

Definition (Steel)

Two conditions p and p^* are μ-F-absolute retaggings if:

- $T^p = T^{p^*}$ and $f^p \restriction F = f^{p^*} \restriction F$;
- h^p and h^{p^*} agree up to μ, i.e.,
  - if $h^p(\sigma) < \mu$, then $h^p(\sigma) = h^{p^*}(\sigma)$;
  - if $h^p(\sigma) \geq \mu$, then $h^{p^*}(\sigma) \geq \mu$ as well.

Lemma

Suppose that ψ only quantifies over ranked sets in some M_F. If p and p^* are μ-F-absolute retaggings, then

$$p \Vdash \psi \qquad \Leftrightarrow \qquad p^* \Vdash \psi.$$

(μ increases with the complexity of ψ.)

A stronger retagging notion for Σ^1_1 formulas

In order to reason about the forcing relation for Σ^1_1 formulas, we require a stronger retagging notion which places restrictions on the locks (and more!)

Definition (G.)

We say that $\mathrm{Ret}_{\leq}(\mu, p, p^*)$ if:

- p and p^* are μ-$\mathrm{dom}(f^p)$-absolute retaggings;
- paths in p and p^* pass through the same nodes of length 1 (as a whole);
- every node which is locked in p^* is also locked in p.

- Unlike Steel's retagging, $\mathrm{Ret}_{\leq}$ is asymmetric.
- The first condition above implies that every path in p is also in p^*.

Roughly speaking, we show that if p forces a Σ^1_1 formula ψ and $\mathrm{Ret}_{\leq}(\mu, p, p^*)$ for some μ, then p^* forces ψ as well.

M_∞ satisfies Δ^1_1-comprehension

Suppose ψ and φ are complementary Σ^1_1 formulas. Goal:

$$\{d \in \omega : \exists q \in G(q \Vdash \psi(\mathbf{d}))\} \in M_\infty.$$

Two key steps:

1. use a boundedness argument to restrict our attention to ψ^μ, which is the result of bounding all set quantifiers in ψ by some $\mu < \omega_1^{CK}$;
2. modify the scope of "$\exists q \in G$" to all conditions q whose tagging functions h^q agree with the actual rank function h^G of T^G up to some $\mu < \omega_1^{CK}$.

We establish these by utilizing the relationships between various notions of retagging and the forcing relation.

M_∞ satisfies Δ^1_1-comprehension

Towards a contradiction, assume that $M_\infty \models \varphi(d)$, yet we are able to find q that satisfies the requirements in the previous slide.

$$
\begin{tikzcd}
 & & \forall n(\psi(n) \leftrightarrow \neg\varphi(n)) & & \\
 & & p \arrow[u, phantom, sloped, "\Vdash"] \arrow[dl] \arrow[dd] \arrow[dr] & & \\
\psi^\mu(\mathbf{d}) & q \arrow[l, phantom, "\Vdash"] \arrow[d] & & r \arrow[r, phantom, "\Vdash"] \arrow[d] & \varphi^\mu(\mathbf{d}) \\
 & q^* \arrow[ul, phantom, sloped, "\nVdash"] \arrow[r, dotted, dash, "\mathrm{Ret}_\leq"] & s^* \arrow[d, phantom, sloped, "\Vdash"] \arrow[r, dotted, dash, "\mathrm{Ret}_\geq"] & r^* \arrow[ur, phantom, sloped, "\nVdash"] & \\
 & & \psi^\mu(\mathbf{d}) \wedge \varphi^\mu(\mathbf{d}) & &
\end{tikzcd}
$$

p and r lie in G, while q “looks like” it lies in G (at least enough so that we can “amalgamate” q and r).

Thank you!