Recent Results in Cohesive Powers

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WDCM-2020
July 24, 2020

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- Motivation: [Dimitrov, 2004, 2008, 2009]
  Cohesive powers used in structural theorems about $\mathcal{L}^*(V_\infty)$.
- Recent Results:
- [Dimitrov, Harizanov, Miller, Mourad, 2014]
  Isomorphisms on non-standard fields and Ash's conjecture
- [Dimitrov, Harizanov, 2015]
  Orbits of maximal vector spaces
- [Dimitrov, Harizanov, Morozov, Shafer, A. Soskova, Vatev, 2019]
  Cohesive Powers of Linear Orders

Definitions from Computability Theory

- $\mathcal{E}$ is the lattice of computably enumerable sets.
- $\mathcal{E}^*$ is the lattice $\mathcal{E}$ modulo finite sets.
- An infinite set $C \subset \omega$ is *cohesive* if for every c.e. set W either $W \cap C$ or $\overline{W} \cap C$ is finite.
- A set M is *maximal* if M is c.e. and $\overline{M}$ is cohesive.
- Theorem: [Friedberg 1958] There are maximal sets.
- B is quasimaximal if it is the intersection of finitely many $(n \geq 1)$ maximal sets.
- $\mathcal{E}^*(B, \uparrow)$ is isomorphic to the Boolean algebra $\mathbf{B_n}$.
- (Soare) If M_1 and M_2 are maximal sets then there is an automorphism g of $\mathcal{E}^*$ such that $g(M_1) = M_2$
- (Soare) If Q_1 and Q_2 are rank-n quasimaximal sets, then there is an automorphism g of $\mathcal{E}^*$ such that $g(Q_1) = Q_2$

The Lattice $\mathcal{L}(V_\infty)$.

- V_∞ is a $\aleph_0$-dimensional vector space over a computable field F.
- The vectors in V_∞ are finite sequences of elements of Q.
- If I is a set of vectors from V_∞,
  then $cl(I)$ denotes the linear span of I.
- $V \subseteq V_\infty$ is computably enumerable if the set of vectors in V is c.e.
- The c.e. subspaces of V_∞ form a lattice, denoted by $\mathcal{L}(V_\infty)$.
- $V_1 \vee V_2 = cl(V_1 \cup V_2)$ and $V_1 \wedge V_2 = V_1 \cap V_2$.

The Lattice $\mathcal{L}^*(V_\infty)$

- $V_1 =^* V_2$ if V_1 and V_2 differ by finite dimension.
- $\mathcal{L}^*(V_\infty)$ is $\mathcal{L}(V_\infty)/=^*$.
- Both $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$ are modular nondistributive lattices.
- There are very few result about the structure of $\mathcal{L}^*(V_\infty)$.
- Conjecture: (Ash) The automorphisms of $\mathcal{L}^*(V_\infty)$ are induced by computable semilinear transformations with finite dimensional kernels and co-finite dimensional images in V_∞.

The map $Cl : \mathcal{E}^* \to \mathcal{L}^*(V_\infty)$

- Let A be a computable basis of V_∞ and let B be a quasimaximal subset of A.
- Let $V = cl(B)$.
- Identify $\mathcal{E}^*$ with the lattice of c.e. subsets of A modulo $=^*$.
- $\mathcal{E}^*(B, \uparrow) \cong \mathbf{B_n}$
- Yet $\mathcal{L}^*(V, \uparrow)$ is not always isomorphic to $\mathbf{B_n}$.

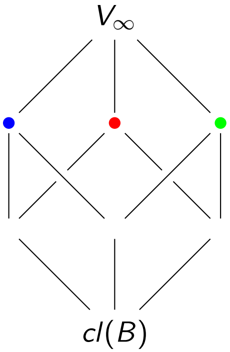
Characterization of $\mathcal{L}^*(V,\uparrow)$

Theorem(s): (Dimitrov 2004, 2008) Let B be a rank-n quasimaximal subset of a computable basis Ω of V_∞ Then $\mathcal{L}^*(cl(B),\uparrow) \cong$

- (1) the Boolean algebra $\mathbf{B_n}$, or
- (2) the lattice $L(n, Q_{\mathbf{a}})$ of all subspaces of an n-dimensional vector space over a certain extension $Q_{\mathbf{a}}$ ($\prod Q$) of the field Q, or
- (3) a finite product of lattices from (1) and (2).

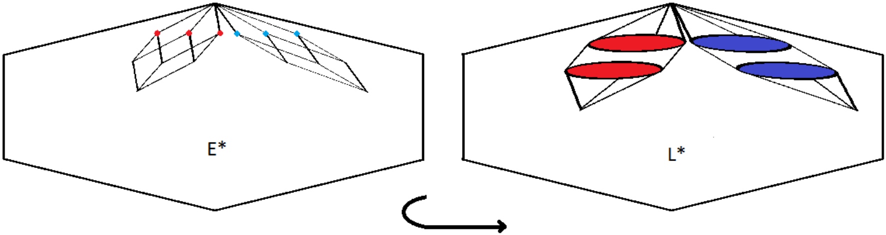
Example 1: $type_{\Omega}(B) = ((1, 1, 1), (\mathbf{a}, \mathbf{b}, \mathbf{c}))$,

- Then $\mathcal{L}^*(cl(B), \uparrow) \cong$
- the Boolean algebra B_3



Example 2: $type_\Omega(B_1) = (3, (\mathbf{a}))$, and $type_\Omega(B_2) = (3, (\mathbf{b}))$

- Assume $\mathbf{a} \neq \mathbf{b}$. Then $Q_\mathbf{a} \ncong Q_\mathbf{b}$
- (By the FTPG) $\mathcal{L}^*(cl(B_1), \uparrow) \ncong \mathcal{L}^*(cl(B_2), \uparrow)$



Cohesive powers of computable structures

Example: The cohesive power $\prod_C Q$ of Q over C

(1) Domain of $\prod_C Q$:
$\{[\varphi]_C \mid \varphi : \omega \to Q$ is a partial computable function, and $C \subseteq^* dom(\varphi)\}$
Here $\varphi_1 =_C \varphi_2$ iff $C \subseteq^* \{x : \varphi_1(x) \downarrow = \varphi_2(x) \downarrow\}$.
$[\varphi]_C$ is the equivalence class of φ with respect to $=_C$

(2) Pointwise operations:
$[\varphi_1]_C + [\varphi_2]_C = [\varphi_1 + \varphi_2]_C$ and $[\varphi_1]_C \cdot [\varphi_2]_C = [\varphi_1 \cdot \varphi_2]_C$

(3) $[0]_C$ and $[1]_C$ are the equivalence classes of the corresponding total constant functions.

[D2008]: Cohesive Powers of Computable Structures

Let $\mathcal{A}$ be a computable structure for a computable language L and with domain A, and let $C \subset \omega$ be a cohesive set. The *cohesive power of $\mathcal{A}$ over C*, denoted by $\prod_C \mathcal{A}$, is a structure $\mathcal{B}$ for L with domain B such that the following holds.

- The set $B = (D/ =_C)$, where $D = \{\varphi \mid \varphi : \omega \to A$ is a partial computable function, and $C \subseteq^* dom(\varphi)\}$.
  For $\varphi_1, \varphi_2 \in D$, we have $\varphi_1 =_C \varphi_2$ iff $C \subseteq^* \{x : \varphi_1(x) \downarrow = \varphi_2(x) \downarrow\}$.
- If $f \in L$ is an n-ary function symbol, then $f^{\mathcal{B}}$is an n-ary function on B such that for every $[\varphi_1], \ldots, [\varphi_n] \in B$, we have $f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi]$, where for every $x \in A$,

$$\varphi(x) \simeq f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)).$$

Definition: Cohesive Powers of Computable Structures

- If $P \in L$ is an m-ary predicate symbol, then $P^{\mathcal{B}}$ is an m-ary relation on B such that for every $[\varphi_1], \ldots, [\varphi_m] \in B$,

$$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \quad \textit{iff} \quad C \subseteq^* \{x \in A \mid P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}.$$

- If $c \in L$ is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the (total) computable function on A with constant value $c^{\mathcal{A}}$.

Related Notions

- Homomorphic images of the semiring of recursive functions have been studied as models of fragments of arithmetics by Fefferman, Scott, and Tennenbaum, Hirschfeld, Wheeler, Lerman, McLaughlin
- Let A be a specific r-cohesive set and let f and g be computable functions.
- Fefferman, Scott, and Tennenbaum defined a structure $\mathcal{R}/\sim_A$
  For recursive f and g let $f \sim_A g$ if $A \subseteq^* \{n : f(n) = g(n)\}$

  The domain of $\mathcal{R}/\sim_A$ consists of the equivalence classes of recursive functions modulo $\sim_A$

Theorem

(Fundamental theorem of cohesive powers)

1. *If* $\tau(y_1, \ldots, y_n)$ *is a term in* L *and* $[\varphi_1], \ldots, [\varphi_n] \in B$*, then* $[\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])]$ *is the equivalence class of a p.c. function such that* $\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])(x) = \tau^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x))$.
2. *If* $\Phi(y_1, \ldots, y_n)$ *is a formula in* L *that is a boolean combination of* Σ_1 *and* Π_1 *formulas and* $[\varphi_1], \ldots, [\varphi_n] \in B$*, then*

$$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n]) \text{ iff } R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}.$$

3. *If* Φ *is a* Π_3 *sentence in* L*, then* $\mathcal{B} \models \Phi$ *implies* $\mathcal{A} \models \Phi$.
4. *If* Φ *is a* Π_2 *(or* Σ_2*) sentence in* L*, then* $\mathcal{B} \models \Phi$ *iff* $\mathcal{A} \models \Phi$.

Observations

If $\mathcal{A}$ is:

- (Dense) Linear Order (Without Endpoints),
- Ring,
- (Algebraically Closed) Field ,
- Lattice,
- (Atomless) Boolean Algebra,

then so is $\prod_{C} \mathcal{A}$.

observations

Let $\mathcal{B} = \prod_{C} \mathcal{A}$ be the cohesive power of $\mathcal{A}$.

For $c \in A$ let $[\varphi_c] \in B$ be the equivalence class of φ_c such that $\varphi_c(i) = c$.

The map $d : A \to B$ such that $d(c) = [\varphi_c]$ is called the canonical embedding of $\mathcal{A}$ into $\mathcal{B}$.

Properties:

- If $\mathcal{A}$ is finite then $\prod_C \mathcal{A} \cong \mathcal{A}$.
- if there is a computable permutation σ of ω such that $\sigma(C_1) =^* C_2$, then $\prod_{C_1} \mathcal{A} \cong \prod_{C_2} \mathcal{A}$.
- If $\mathcal{A}_1 \cong \mathcal{A}_2$ are computably isomorphic then $\prod_C \mathcal{A}_1 \cong \prod_C \mathcal{A}_2$.
- Let C be be co-r.e. Then for every $[\varphi] \in \prod_C \mathcal{A}$ there is a computable function f such that $[f] = [\varphi]$.

  $$f(n) = \begin{cases} \varphi(n) & \text{if } \varphi(n)\downarrow \text{ first,} \\ a & \text{if n is enumerated into } \overline{C} \text{ first.} \end{cases}$$

Limited Los for $\prod_{\overline{M}} \mathbb{N} \cong \mathcal{R}/\sim_{\overline{M}}$

- If M is a maximal set, then $\prod_{\overline{M}} \mathbb{N} \cong \mathcal{R}/\sim_{\overline{M}}$
- Feferman-Scott-Tennenbaum [1959]:
  $\mathcal{R}/\sim_A$ is a model of only a fragment of First-Order-Arithmetics.

  $N \models \forall x \exists s \forall e \leq x[\varphi_e(x)\downarrow \rightarrow \varphi_{e,s}(x)\downarrow]$

  but

  $\mathcal{R}/\sim_A \not\models \forall x \exists s \forall e \leq x[\varphi_e(x)\downarrow \rightarrow \varphi_{e,s}(x)\downarrow]$

Results by J.Hirschfeld, Lerman, McLaughlin, Wheeler

Theorem: (Lerman 1975)

- (1) If $A_1 \equiv_m A_2$ are r-maximal sets, then $R/\overline{A_1} \cong R/\overline{A_2}$.
- (2) If M_1 and M_2 are maximal sets of different m-degrees, then $R/\overline{M_1}$ and $R/\overline{M_2}$ are not even elementary equivalent.

Theorem: (Hirschfeld and Wheeler 1975), (McLaughlin 1990)

- (3) $R/\overline{M}$ is rigid

Dimitrov, Harizanov, Miller, Mourad 2014

1. If $M_1 \equiv_m M_2$, then $\prod_{M_1} \mathbb{Q} \cong \prod_{M_2} \mathbb{Q}$.
2. If $M_1 \not\equiv_m M_2$, then $\prod_{M_1} \mathbb{Q} \not\cong \prod_{M_2} \mathbb{Q}$.
3. $\prod_{M} \mathbb{Z}$ is rigid.
4. $\prod_{M} \mathbb{Q}$ is rigid.

- $Q_{\mathbf{a}}$ is the cohesive power of Q with respect to the complement of a maximal set with m-degree **a**

- (4) $Q_{\mathbf{a}}$ is rigid.

- (2) If $\mathbf{a} \neq \mathbf{b}$, then $Q_{\mathbf{a}}$ is not isomorphic to $Q_{\mathbf{b}}$.

Definability of Z in Q

- J. Robinson (1949): Z is $\forall\exists\forall$ definable in Q.
- Poonen (2009): Z is $\exists\forall$ definable in Q.
- Koenigsmann (2015): Z is $\forall$ definable in Q.

Definability of N in Q

- N is definable in Z :

  $y \in N \Leftrightarrow \exists z_1 \ldots \exists z_4 [y = z_1^2 + z_2^2 + z_3^2 + z_4^2]$
- Corollary: N definable in Q:
  $x \in N \Leftrightarrow \exists y_1 \ldots \exists y_4 [\bigwedge_{i \leq 4} y_i \in Z \wedge x = y_1^2 + y_2^2 + y_3^2 + y_4^2]$

Dimitrov, Harizanov, Orbits of Maximal Vector Spaces, Algebra and Logic, 2015.

- There is an automorphism σ of $\mathcal{L}^*(V_\infty)$ such that $\sigma(cl(B_1)) = cl(B_2)$

  iff

- $type(B_1) = type(B_2)$.

Example 1: $type_{\Omega}(B) = ((1,1,1), (\mathbf{a}, \mathbf{b}, \mathbf{c}))$,

- Then $\mathcal{L}^*(cl(B), \uparrow) \cong$
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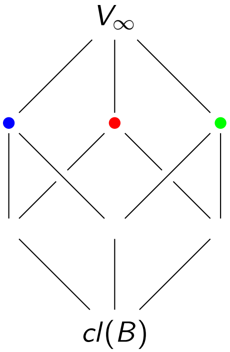

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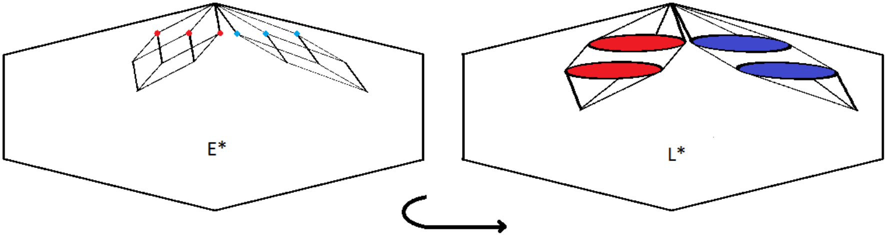

Main Results: Cohesive Powers of Linear Orders, Dimitrov, Harizanov, Morozov, Shafer, A. Soskova, Vatev, 2019

- If $\mathcal{A}$ is a computable presentation of the linear order $\mathbb{N}$ with a computable successor function, then

$$\prod_{C} \mathcal{A} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}.$$

- There is a computable presentation $\mathcal{B}$ of the linear order $\mathbb{N}$ with a non-computable successor function, s.t.

$$\prod_{C} \mathcal{B} \cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$$

Main Results: Cohesive Powers of Linear Orders, Dimitrov, Harizanov, Morozov, Shafer, A. Soskova, Vatev, 2019

- There is a computable presentation $\mathcal{D}$ of the linear order $\mathbb{N}$, s.t.

$$\prod_C \mathcal{D} \not\cong \mathbb{N} + \mathbb{Q} \times \mathbb{Z}$$

In fact, there is an element of $\prod_C \mathcal{D}$ that has no immediate successor.

- Let φ be a $\prod_3^0$ sentence that states that every element has an immediate sucessor. Then

$$\mathbb{N} \models \varphi \text{ while } \prod_C \mathcal{D} \not\models \varphi$$

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Thank you