Computable groups and computable group orderings

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Linear orderings on groups

Definition of *bi-orderable* groups

Let G be a group and $<$ be a linear order on G. G is said to be *bi-orderable* with respect to $<$ if for each $g, h, x \in G$ if $g \leq h$, then

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A naturally associated concept with group orders is *positive cone* that can be defined as follows:

$$PC(G, <) := \{g \in G \mid g > 1\}$$

Some remarks

- The bi-orderings on groups gained popularity after seminal works of Dedeking, Hölder, and Hilbert, where they were considering bi-orderings in a broad algebraic context;
- In more abstract group theoretical context bi-orderable groups where intensively studied starting from 1940's by Levi, B.Neumann, and others.
- Left-orderable groups have more modern origin. However, due to their natural occurrence in groups' classes with interesting geometric, topological, and dynamical properties, in recent years, they gained broad popularity. For example, we have

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Theorem

A countable group G is left-orderable if and only if it embeds into $Homeo^+(\mathbb{R})$, the group of orientation preserving homeomorphisms of $\mathbb{R}$.

Computable groups-1

Interactions between *combinatorial group theory* and *computability theory* has a long history that goes back to the seminal work of Max Dehn from 1911, where he introduced word, conjugacy, and isomorphism problems in *finitely generated* groups. The highest points in this area are the theorems of Higman and Boone-Higman that correspondingly state:

- (Higman, 1961) A given finitely generated group has a recursive presentation if and only if it *embeds* into a finitely presented group;
- (Boone-Higman, 1974) A finitely generated group has decidable word problem if and only if it embeds into a simple subgroup of a finitely presented group.

Computable groups-2

Seminal works of Frölich-Shepherdson, Rabin, and Mal'cev, done in 1950's and 1960's, significantly extended the scope of algebraic structures the computability properties of which were of interest. In particular, the analog of groups with decidable word problem for countable (but not necessarily f.g.) groups was introduced, independently, by Rabin and by Mal'cev.

Definition (Rabin, 1960; Mal'cev, 1961)

A presentation $G = \langle X \mid R \rangle$ of a countable group is called *computably enumerated* if the sets X and $R \subseteq (X \cup X^{-1})^*$ are computably enumerated. It is said that $G = \langle X \rangle$ is a *computable group* with respect to the computably enumerated generating set X if the set

$$\{u \in (X \cup X^{-1})^* \mid u =_G 1\}$$

is computable.

Computable orders on groups

In the context of computability theory on algebraic structures, it is very natural to consider computability properties of structures associated with ordering on groups. In particular, in 1986, Downey and Kurtz initiated a systematic study of computability theory of positive cones of ordered groups.

Definition (Computable orders)

Let G be a (countable) group and $<$ be a linear order on it. Then, $<$ is said to be computable with respect to the given presentation $G = \langle X \mid R \rangle$ if

- G is computable with respect to that presentation, and
- $PS(G, <)$ is computably enumerable.

In other words, X is computably enumerated and for any $w \in (X \cup X^{-1})^*$ one can algorithmically realize whether $w >_G 1$, $w =_G 1$, or $w <_G 1$.

Downey-Kurtz's question

G is said to be *computably (bi- or left-) orderable* if it possesses a (bi- or left-) order $<$ and a presentation with respect to which $<$ is computable.

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Question of Downey and Kurtz, 1999

Is every computable orderable group isomorphic to computably orderable group?

For abelian groups, a positive answer to the question of Downey and Kurtz was obtained by Reed Solomon in 2002.

Theorem (R. Solomon, 2002)

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In case of left-orderable groups, Harrison-Trainor showed that, in general, the answer to the question is negative.

Theorem (Harrison-Trainor, 2018)

There exists a computable left-orderable group G that does not possess a computable left-order with respect to any presentation of G.

Harrison-Trainor's result extends to the general case of bi-orderable groups in a stronger form.

Theorem (D., 2019)

There exists a two-generated bi-orderable computable group G that does not embed in any countable group with a computable left-order. Moreover, G can be chosen to be a solvable group of derived length 3.

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Question. Does there exist a computable bi-orderable metabelian group that does not possess a computable bi-order?

Theorem (D., 2015, 2019)

Let $H = \langle X \rangle$ be a group with countable generating set $X = \{x_1, x_2, \ldots\}$. Then there exists an embedding $\Phi_X : H \hookrightarrow G$ into a two-generated group $G = \langle f, s \rangle$ such that the following holds.

1. *There exists a computable map $\phi_X : i \mapsto \{f^{\pm 1}, s^{\pm 1}\}^*$ such that $\phi_X(i)$ represents the element $\Phi_X(x_i)$ in G;*

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2. G *has a computable presentation if and only if* H *has a computable presentation with respect to the generating set* X*;*

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2. *G has a computable presentation if and only if H has a computable presentation with respect to the generating set X;*
3. *G has decidable word problem if and only if H is computable with respect to the generating set X;*

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3. *G has decidable word problem if and only if H is computable with respect to the generating set X;*
4. *If H is a computable group with respect to the generating set X, then the membership problem for the subgroup $\Phi_X(H) \leq G$ is decidable, i.e. there exists an algorithm that for any $g \in G$ decides whether or not $g \in \Phi_X(H)$;*

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1. *There exists a computable map $\phi_X : i \mapsto \{f^{\pm 1}, s^{\pm 1}\}^*$ such that $\phi_X(i)$ represents the element $\Phi_X(x_i)$ in G;*
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5. *If H is left- or bi- orderable, then so is G. Moreover, if with respect to the generating set X there is a computable order on H, then G has a computable order as well.*

To obtain the answer to the question of Downey-Kurtz in the stronger form, one can apply the embedding theorem on the group $H = \bigoplus_{i=1}^{\infty} \mathbb{Z}_i$ that is given by a specific presentation

$$\langle a_i, i = 1, 2, \ldots \mid [a_i, a_j] = 1, a_{2n_i} = a_{2n_i-1}^{p_i}, a_{2m_i} = a_{2m_i-1}^{-p_i}, i, j \in \mathbb{N} \rangle,$$

where $\{p_1, p_2, \ldots\}$ is the set of primes in its natural order, and $\mathcal{M} = \{m_1, m_2, \ldots\}$ and $\mathcal{N} = \{n_1, n_2, \ldots\}$ are disjoint pair of recursively enumerable and recursively inseparable sets of natural numbers. (Up to my knowledge, pairs of sets with such properties first appeared in applications by Smullyan in 1958.)

Some corollaries

Corollary (A characterization of computable groups, D., '19)

A countable group H *is computable if and only if it is a subgroup of a finitely-generated group with decidable word problem such that the membership problem for the subgroup* H *is decidable in the large group.*

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Corollary (A characterization of computable orders, D., '19)

A countable group H has a computable left- or bi- ordering if and only if it is a subgroup of a finitely-generated group with computable left- or bi- order, respectively, such that the membership problem for the subgroup H is decidable in the large group. Moreover, for any fixed computable order on H we can assume that the large group continues the order on H.

Possible further directions in the study of computability properties of orderable groups is the study of positive cones from the perspective of computational complexity; Turing degrees; and formal languages.

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For example,

- For a fixed group, how independent can be computational complexity properties of positives cones with respect to different orderings? (Note that, by a theorem of Linnell, a countable left-orderable group has either finite number or continuum many different left-orderings.)
- Can one characterize finitely generated bi-orderable groups with a positive cone being a regular or context-free language? (I.e. obtain an Anisimov or Mueller-Schupp type theorem with respect to the formal language of positive cones.)
- How arbitrary can be Turing degrees of positive cones?