Complexity of fixed-point selection functions

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Kleene's Recursion Theorem

Theorem (Kleene'1937)

For every computable function f there exists an n, called a fixed point of f, such that

$$\Phi_{f(n)} \simeq \Phi_n.$$

(i.e. f has a fixed point n).

Here Φ_m the m-th partial computable function according to some fixed effective listing of all such functions.
Let W_m denotes $\mathrm{d}om(\Phi_m)$.

Completeness Criterion

Theorem (A.'1977)

Let A be a c.e. set such that $\emptyset' \not\leq_T A$. Then any unary function $f \leq_T A$ have a fixed-point:

$$(\exists x)(\Phi_{f(x)} \simeq \Phi_x).$$

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Corollary

Let A be a c.e. set. Then A is Turing complete if and only if there is a unary function $f \leq_T A$ such that

$$(\forall x)(\Phi_x \not\simeq \Phi_{f(x)}).$$

First Generalization: Almost fixed-points and Σ_2^0-sets.

Let $A \sim^* B \leftrightarrow A \cup F_1 = B \cup F_2$ for some finite sets F_1 and F_2.

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Theorem.

a) for every $f : \omega \mapsto \omega$, if $f \leq_T \emptyset'$ then $(\exists e)[W_e \sim^* W_{f(e)}]$;

b) for any set $A \in \Sigma_2^0$,

$$\emptyset' \leq_T A \Rightarrow [A \equiv_T \emptyset'' \Leftrightarrow (\exists f \leq_T A)(\forall e)\neg[W_e \sim^* W_{f(e)}]];$$

Second Generalization: $\equiv_T$-fixed-points and Σ^0_3-sets.

Theorem.

a) for every $f : \omega \mapsto \omega$, if $f \leq_T \emptyset''$ then $(\exists e)[W_e \equiv_T W_{f(e)}]$;

b) for any set $A \in \Sigma^0_3$,

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Theorem.(Jockusch, Lerman, Soare and Solovay'1989)

a) for every $f : \omega \mapsto \omega$ and $n \geq 0$,
$f \leq_T \emptyset^{(n+3)} \rightarrow (\exists e)[W_e \sim_{n+3} W_{f(e)}]$;

b) for any set $A \in \Sigma^0_{n+4}$,

$$\emptyset^{(n+3)} \leq_T A \Rightarrow [A \equiv_T \emptyset^{(n+4)} \Leftrightarrow (\exists f \leq_T A)(\forall e)\neg[W_e \sim_{n+3} W_{f(e)}]];$$

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Theorem.(Jockusch, Lerman, Soare and Solovay'1989)

a) for every $f : \omega \mapsto \omega$, if $f \leq_T \emptyset^{(\omega)} \to (\exists e)[W_e \sim_\omega W_{f(e)}]$;

b) for any set $A \in \Sigma^0_{\omega+1}$,

$$\emptyset^{(\omega)} \leq_T A \Rightarrow [A \equiv_T \emptyset^{(\omega+1)} \Leftrightarrow (\exists f \leq_T A)(\forall e)\neg[W_e \sim_\omega W_{f(e)}]].$$

Further considerations

$m-$, $tt-$, $wtt-$ reducibilities (Arslanov),
$Q-$ reducibility (Baturshin)

Further considerations.

Definition.
Let $R \subset \omega \times 2^\omega$ be a relation which generates a reflexive and transitive relation $\leq_R$:

$$A \leq_R B \rightleftharpoons (\forall x)[x \in A \leftrightarrow R(h(x), B)]$$

for some computable function h.

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and $\{x : R(h(x), A)\}$ c.e., $(\forall x)[\Phi_x \not\simeq \Phi_{f(x)}]$.

Uniformity of the Recursion Theorem

Let $h(n, x)$ be a computable function of two arguments. Then for every fixed $n, \lambda x.h(n, x)$ have a fixed point (by Kleene's Recursion Theorem): $(\forall n)(\exists x)(\Phi_x \simeq \Phi_{h(n,x)})$.

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Moreover, the recursion theorem holds uniformly:

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Theorem (Kleene)

For every partial computable function $\psi(n,x)$ there is a computable function $f(n)$ such that for all n

$$\psi(n,f(n))\downarrow \rightarrow \Phi_{f(n)} \simeq \Phi_{\psi(n,f(n))}.$$

On the Uniformity of my Generalization

Theorem

Let A be a c.e. set such that $\emptyset' \not\leq_T A$. Then any total unary function $f \leq_T A$ have a fixed-point:

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Let A be a c.e. set such that $\emptyset' \not\leq_T A$. Then any total unary function $f \leq_T A$ have a fixed-point:

$$(\exists x)(\Phi_{f(x)} \simeq \Phi_x).$$

Let the set A from the previous theorem is fixed. Then each total function in the sequence $\Phi_0^A, \Phi_1^A, \Phi_2^A, \ldots$ have a fixed point. It is natural to ask: is there a computable function $f(e)$ such that for any fixed e $f(e)$ is a fixed-point for the function $\Phi_e^A(x)$, if it is total:

$$\Phi_e^A \ \text{ total } \ \rightarrow \Phi_{f(e)} \simeq \Phi_{\Phi_e^A(f(e))}$$

Question:

A natural question arises:

whether this uniformity property holds for the above generalizations of the Recursion Theorem. And if not, that is, if there is no computable procedure for finding such fixed points, then in what smallest degrees can such procedures be?

Sebastiaan Terwijn was the first who interested in this question, he also received first results in this direction. Later, as a result of correspondence with him, I also became involved in this problem.

Fixed point selection functions

Definition (for binary functions).

Let $h(n, x)$ be a function of two arguments, and each unary function $\lambda x.h(n, x)$ have a fixed-point x_n: $\Phi_{x_n} \simeq \Phi_{h(n,x_n)}$). Let's call the function $f(n) = x_n$ the *fixed point selection function for* the sequence $\{h(n, x)\}_{n\in\omega}$ (FP-selection function, for short).

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Definition (for subsets of ω)

Suppose that A is a c.e. set such that any total unary function $h \leq_T A$ have a fixed point, and let f be a unary partial function such that for every e, if the unary function $\{\lambda x.\Phi_e^A(x)\}$ is total then $f(e)$ defined and $\Phi_{f(e)} \simeq \Phi_{\Phi_e^A(f(e))}$.
We will say that f is a fixed-point selection function for the set A.

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Suppose that every computable in A unary function have a fixed point and let h be a binary function computable in A: $h(x,y) = \Phi_e^A(x,y)$. Then for each fixed y, the function $\lambda x.h(x,y)$ have a fixed point (which obviously depends on y: $(\exists x_y(\Phi_{x_y} \simeq \Phi_{h(x_y,y)})$.
Let f be a fixed-point selection function for the sequence $\{h(x,0), h(x,1), \ldots, \}$:

$$(\forall x)(\Phi_{f(x)} \simeq \Phi_{h(x,f(x))})$$

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Let f be a fixed-point selection function for the sequence $\{h(x, 0), h(x, 1), \ldots, \}$:

$$(\forall x)(\Phi_{f(x)} \simeq \Phi_{h(x, f(x))})$$

In general, the degree of the FP-selection function f depends on the selected Turing functional $\Phi_e^A (= h)$.

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Now consider my completeness criterion for c.e. sets.

Theorem (A.'1977)

Let A be an *incomplete* c.e. set ($\emptyset' \not\leq_T A$). Then A have a fixed-point selection function; i.e. there is a partial function f such that

$$(\forall e)[\Phi_e^A \text{ total} \rightarrow \mathrm{f(e)}\downarrow \ \& \ \Phi_{\Phi_e^A(\mathrm{f(e)})} \simeq \Phi_{\mathrm{f(e)}}]$$

Theorem

A c.e. set A is incomplete if and only if it have a total fixed-point selection function $f(e)$: for all e,

$$\Phi_e^A(f(e)) \downarrow \to \Phi_{f(e)} \simeq \Phi_{\Phi_e^A(f(e))}$$

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Theorem

For any c.e. set A, if $\emptyset' \not\leq_T A$, then A have a total fixed-point selection function $f \leq_T A$.

Theorem (Terwijn'2018)

There exist a low c.e. set A ($A' \equiv_T \emptyset'$) and a binary function $h \leq_T A$ such that for every computable f, there exists n with $\Phi_{f(n)} \not\simeq \Phi_{h(n,f(n))}$.

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Theorem

Let A be an arbitrary non-computable c.e. set and $\emptyset' \not\leq_T A$.
a) There is a binary function $h, h \equiv_T A$, which have no computable FP-selection functions;

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Theorem

Let A be an arbitrary non-computable c.e. set and $\emptyset' \not\leq_T A$.

a) There is a binary function $h, h \equiv_T A$, which have no computable FP-selection functions;

b) There is a binary function $\psi(e,x), \psi \leq_T A$, such that for any binary function $h = \Phi_e^A$, $(\forall n)\{\Phi_{\psi(e,n)} \simeq \Phi_{h(n,\psi(e,n))}\}$.

Sketch of the proof for a).

Let A be a non-computable c.e. set, $\emptyset' \not\leq_T A$. We build a total binary function $h \leq_T A$ in an ω-sequence of cycles meeting following requirements:

$$\mathcal{R}_e: \qquad f = \Phi_e \text{ total} \rightarrow (\exists n)[W_{f(n)} \neq W_{h(n,f(n))}]$$

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For each $\mathcal{R}_e$ we at a cycle k pick as a witness an integer n from a column assigned for $\mathcal{R}_e$ and wait for $\Phi_e(n)(= f(n)) \downarrow$ at a stage s. If $W_{f(n),s} \neq \emptyset$ then define $h(n, f(n))$ so that $W_{h(n,f(n))} = \emptyset$ forever, satisfying $\mathcal{R}_e$. If not, define $h(n, f(n))$ with a big A-use so that $W_{h(n,f(n))} \neq \emptyset$, and wait for a later stage $t > s$ when $W_{f(n),t}$ again becomes nonempty. Now we have to change $h(n, f(n))$ to make $W_{h(n,f(n))}$ empty. For this we wait for A-change below its use at stage s, and open a new cycle $k + 1$ with a new bigger witness, and proceed by the same way.

Since A is non-computable, there must be a cycle when we have a success.

It is easy achieve deg $(A) =$deg (h), proceeding the previous construction for even arguments of h, and coding the set A in its odd arguments.

For part $b)$ adapt my proof of the completeness criterion.

Questions

What can we say about the complexity of the FP-selection functions?

In the general setting: is there a relationship between the complexity (in several senses) of fixed point selection functions for a set A and the complexity of the set A itself (also in several senses).

Open Questions

Question 1.

Can non-computable c.e. set A, $\emptyset' \not\leq_T A$, have a computable FP-selection function?

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Conjecture.

For any c.e. set A, existence of a computable FP-selection function entails computability of A.

Theorem

For any non-computable c.e. set A there is a non-computable binary function $\lambda e, x.h(e, x)$, $h \leq_T A$ such that the sequence $h(0, x), h(1, x), \ldots$ have a computable FP-selection function.

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Question 3. Is the following true:

let ψ be a non-computable binary function, $\deg(\psi) < \mathbf{0}'$ and $\mathbf{a}$ be a c. e. degree such that $\deg(\psi) < \mathbf{a} < \mathbf{0}'$. Then is there a c. e. set $A \in \mathbf{a}$ such that ψ is a fixed-point selection function for A?

Remark.

I think that FP-selection functions of non-computable and non-complete c.e. sets coincide with the degrees of these sets.

To prove that we can try to construct a suitable binary Turing reducible to this set A (say) function, which codes the information on the membership in A. Then use this property to try to Turing reduce the set A to its FP-selection function.

But I can not yet to convert this idea into a strict proof.

Question $3'$. Is the following true:

Let $\mathbf{0} < \mathbf{b} < \mathbf{a} < \mathbf{0}'$ be c.e. degrees. Then

a) For every c. e. set $A \in \mathbf{a}$ there is binary function $\psi \in \mathbf{b}$ such that ψ is a fixed-point selection function for A.
b) For every binary total function $\psi \in \mathbf{b}$ there is a c.e. set $A \in \mathbf{a}$ such that ψ is a fixed-point selection function for A.

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let ψ be a non-computable unary function, $\deg(\psi) < \mathbf{0}'$ and $\mathbf{a}$ be a c. e. degree such that $\deg(\psi) < \mathbf{a} < \mathbf{0}'$. Then there is a binary function $h(e, x)$ such that deg $(h) = \mathbf{a}$ and ψ is a fixed-point selection function for h: $\Phi_{\psi(e)} \simeq \Phi_{h(\psi(e))}$.

m-and tt-completeness criterions

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Definition.

Let A be a c.e. set. By definition a function $f : \omega \mapsto \omega$ is m-(tt-)reducible to A, if there are computable functions a, b and h such that

$$f(x) = \begin{cases} a(x), & \text{if } h(x) \in A; \\ b(x), & \text{if h(x)} \notin A. \end{cases}$$

and, accordingly,

$$f(x) = \begin{cases} a(x), & \text{if } A \models \tau_{h(x)}; \\ b(x), & \text{if A} \not\models \tau_{h(x)}. \end{cases}$$

m-and tt-completeness criterions

Theorem (m-completeness criterion; A.'1987)

Let A be a c.e. set. Then A is m-complete if and only if there is a function $f \leq_m A$ without fixed-points: $(\forall x)\{\Phi_{f(x)} \not\simeq \Phi_x\}$.

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Theorem (tt-completeness criterion; A.'1987)

Let A be a c.e. set. Then A is tt-complete if and only if there is a function $f \leq_{tt} A$ without fixed-points: $(\forall x)\{\Phi_{f(x)} \not\simeq \Phi_x\}$.

Let $\{\Omega_i, \Psi_j, \Theta_k\}_{e=\langle i,j,k\rangle \in \omega}$ be an enumeration of partial computable functions Ω, Ψ and Θ. Define $\Phi_e(x) = \Omega_i(x)$, if $\Theta_k(x) \downarrow \& \Theta_k(x) \in A \;\; (A \models \tau_{\Theta_k(x)})$, and $\Phi_e(x) = \Psi_j(x)$, if $\Theta_k(x) \downarrow \& \Theta_k(x) \notin A \;\; (A \not\models \tau_{\Theta_k(x)})$.

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The enumeration $\{\Phi_e\}_{e\in\omega}$ contains the list of m-(tt-) reducible to A functions: by definition, the function Φ_e is m-(tt-) reducible to A, if it is total.

Theorem

There exists a non-computable c.e. set A such that for any total binary function $h \leq_m A$ (for any total binary function $h \leq_{tt} A$),
- if there is a computable $\leq_m$-fixed-point selection function f for the sequence $\{h(x, n) : n \in \omega\}$, then h is computable.
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- there is a noncomputable total binary function $h \leq_T A$ such that it have a computable fixed-point selection function.

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Theorem

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Connection with Kolmogorov complexity

Let M be a universal Turing machine which is optimal for Turing machines $U : 2^{<\omega} \to 2^{<\omega}$:

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so $K_M(\sigma)$ is the length of the shortest M-description of σ.

We write $K(\sigma)$ for $K_M(\sigma)$ and call it the Plain (Kolmogorov) complexity of $\sigma \in 2^{<\omega}$.

Definition (Kjos-Hanssen, Merkle, and Stephan'2011)

a) A set A is *complex* if there is an nondecreasing and unbounded computable function g such that for all n,

$$K(A\lceil n) \geq g(n).$$

Here $A\lceil n$ is the finite binary sequence $A(0), A(1), \ldots A(n-1)$.

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Here $A\lceil n$ is the finite binary sequence $A(0), A(1), \ldots A(n-1)$.

b) A set A is *autocomplex* if there is an nondecreasing and unbounded function $g \leq_T A$ such that $K(A\lceil n) \geq g(n)$ for all y.

Theorem (Kjos-Hanssen, Merkle, and Stephan'2011)

Let A be a c.e. set. If there is an unbounded function $g \leq_T A$ which bounds from below initial segments of A (i. e. $K(A\lceil n) \geq g(n)$ for all n), then there are no FP-selection functions $f \leq_T A$ for A, and vice-versa.

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Let A be a c.e. set. If there is an unbounded function $g \leq_T A$ which bounds from below initial segments of A (i. e. $K(A \lceil n) \geq g(n)$ for all n), then there are no FP-selection functions $f \leq_T A$ for A, and vice-versa.

Corollary.

Let A be a c.e. set. Then there is a FP-selection function $f \leq_T A$ for A if and only if A is non autocomplex.

Definition (Ishmukhametov'1999)

A set A is *c.e. traceable* (*weak computable*) if there is a computable function f such that for every function $g \leq_T A$, there is a computable function h such that $|W_{h(n)}| \leq f(n)$ for every n, and $g(n) \in W_{h(n)}$ for all n ($W_{h(n)}$ is a *trace* for $g(n)$)

Definition (Lewis-Pye'2007)

A set A is *weakly c.e. traceable* if there is a computable function f such that for every function $g \leq_T A$, there is a computable function $h(n)$ such that $|W_{h(n)}| \leq f(n)$ for every n,
and $g(n) \in W_{h(n)}$ for infinitely many n
(and similarly $W_{h(n)}$ is a weak trace for $g(n)$).

Theorem (Lewis-Pye'2007)

For every c.e. set A the following are equivalent:

1. A have a fixed-point selection function $h \leq_T A$;
2. A is weakly c.e. traceable.

Precomplete Numberings

Definition

A numbering of a set S is a surjection $\gamma : \omega \to S$.
Given γ, define an equivalence relation on ω by $n \sim_\gamma m$ if $\gamma(n) = \gamma(m)$.
A numbering γ is precomplete if for every partial computable unary function ψ there exists a computable unary f such that for every n,

$$\psi(n)\downarrow \Rightarrow f(n) \sim_\gamma \psi(n)$$

(f totalizes ψ modulo $\sim_\gamma$.)

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A precomplete numbering γ is complete if there is a special element $a \in \omega$ such that also $f(n) \sim_\gamma a$ for every n with $\psi(n) \uparrow$.

Precomplete Numberings

Theorem (Ershov'1975)

Let γ be a precomplete numbering, and let h be a computable function. Then h has a fixed point modulo $\sim_\gamma$, i. e. there exists $n \in \omega$ such that $h(n) \sim_\gamma n$.

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Theorem (Ershov's Recursion Theorem with Parameters)

Let γ be a precomplete numbering, and let $h(x, n)$ be a binary computable function. Then there exists a computable function f such that for every n, $f(n) \sim_\gamma h(f(n), n)$.

Precomplete Numberings

Theorem (Completeness Criterion for Precomplete Numberings; Barendregt, Terwijn,2019)

Let γ be a precomplete numbering, and let $A <_T \emptyset'$ be a incomplete c.e. set. If g an A-computable function, then g has a fixed point modulo γ, i. e. there exists a $n \in \omega$ such that $g(n) \sim_\gamma n$.

Precomplete Numberings

Theorem

Let γ be a precomplete numbering.
a) If A is an incomplete c.e. set, then there is a non-computable binary function $h \leq_T A$ such that the sequence $h(0,x), h(1,x), \ldots$ have a computable FP-selection function f: for every n, $f(n) \sim_\gamma h(f(n), n)$.

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b) For any non-computable c.e. set A there is a binary function $h \leq_T A$ such that for every computable f, there exists n with $f(n) \not\sim_\gamma h(n, f(n))$.